QUARTERLY OF APPLIED MATHEMATICS

Vol. IX April, 1951 No. 1

A CONVENIENT GENERAL SOLUTION OF THE CONFLUENT HYPERGEOMETRIC EQUATION, ANALYTIC AND NUMERICAL DEVELOPMENT*

BY

T. S. KUHN**

Harvard University

1. Introduction. The standard forms for the general solution of the Confluent Hypergeometric Equation prove too unwieldy for application to many physical problems, particularly in the field of Quantum Mechanics. The two standard power series solutions,[1] $M_{n,\pm m}(y)$, reduce to a single regular polynomial solution whenever $2m$ is an integer (the standard case for quantum mechanics), and in this case the two integral solutions[2], $W_{\pm n,m}$ $(\pm y)$, must be computed with an asymptotic expansion which is cumbersome for most physically interesting values of y.

The utility of all of these solutions is further limited because their form necessitates undertaking a complete recomputation for every physically significant value of n. This makes the labor of computation almost prohibitive in the physically important case where both n and y must be treated as continuous variables.

The possibility of achieving a more manageable form of the solutions was first indicated by the work of Wannier[3] and Jastrow.[4] Wannier showed that in theory the function $M_{n,m}(y)$ could be developed as a series in descending powers of n with coefficients given in terms of Bessel functions. Jastrow actually exhibited the first two terms of an asymptotically similar series for the solution $W_{n,m}(y)$. This paper completes the above treatments by producing analytically a *general* solution of the differential equation as a power series in $1/n^2$ with coefficients readily calculable in terms of known functions. This treatment differs from those noted above not only in the generality of its results, but also in the ease with which successive terms of the series may be explicitly generated. The method employed here makes it possible to exhibit the two particular solutions of the equation which go to zero as $y \to 0$ and as $y \to +\infty$ and to relate these analytically to the earlier solutions $W_{n,m}(y)$ and $M_{n,m}(y)$. Finally this paper will exhibit analytic and numerical values for the coefficients of several of the series of greatest physical interest.

2. The general series solution. For physical applications Whittaker's standard form of the Confluent Hypergeometric Equation,

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*Received January 25, 1950.

**Junior Fellow, Society of Fellows.

[1]E. T. Whittaker and G. N. Watson, *Modern analysis*, (Cambridge University Press, 1940), fourth edition, §16.1.

[2]*Ibid.* §16.12.

[3]G. H. Wannier, Phys. Rev. **64**, p. 358 (1943).

[4]R. Jastrow, Phys. Rev. **73**, p. 60 (1948).

$$\frac{d^2U}{dy^2} + \left[-\frac{1}{4} + \frac{n}{y} + \frac{1/4 - m^2}{y^2}\right]U = 0, \tag{1}$$

is conveniently transformed by the substitutions

$$r = \frac{1}{2}ny; \qquad m = l + \frac{1}{2} \tag{2}$$

to the form

$$\frac{d^2U^{(l,n)}}{dr^2} + \left[-\frac{1}{n^2} + \frac{2}{r} - \frac{l(l+1)}{r^2}\right]U^{(l,n)} = 0, \tag{3}$$

which, with $\epsilon = -1/n^2$, is just the hydrogenic radial wave equation in Rydberg units. The form (3) will be taken as standard in this paper.

The further substitutions

$$z = (8r)^{1/2}; \qquad U^{(l,n)} = \frac{1}{2}zV^{(l,n)} \tag{4}$$

reduce (3) to the form

$$\nabla_l V^{(l,n)} - n^{-2}\left(\frac{1}{2}z\right)^4 V^{(l,n)} = 0, \tag{5}$$

where ∇_l is the Bessel operator of index $2l + 1$, i.e.

$$\nabla_l \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - (2l+1)^2. \tag{6}$$

It will be assumed here and proven in the Appendix that the general solution of (5) may be written in the form of a power series in $1/n^2$, i.e.

$$V^{(l,n)}(z) = \sum_{k=0}^{\infty} n^{-2k}V_k^{(l)}(z), \tag{7}$$

where the functions $V_k^{(l)}(z)$ are analytic functions of z and the series (7) converges absolutely and uniformly for all real l and for all real n in the region $|n| \geq n_0$, n_0 being an arbitrary positive number. Corresponding solutions of (3) may then be written in the form

$$U^{(l,n)}(z) = \sum_{k=0}^{\infty} n^{-2k}U_k^{(l)}(z), \tag{8}$$

in which the $U_k^{(l)}(z)$ are given in terms of the $V_k^{(l)}(z)$ by the equation

$$U_k^{(l)}(z) = \frac{1}{2}zV_k^{(l)}(z). \tag{9}$$

Since (7) converges uniformly and absolutely it may be inserted in the differential equation (5), and differentiated term by term. The terms may then be rearranged and the coefficients of the various powers of $1/n^2$ equated to zero, which is necessary and sufficient to make (7) a solution of (5). This procedure yields an infinite set of simultaneous differential equations for the coefficients $V_k^{(l)}(z)$:

$$\nabla_l V_0^{(l)} = 0 \tag{10a}$$

$$\nabla_l V_k^{(l)} = \left(\frac{1}{2}z\right)^4 V_{k-1}^{(l)}. \tag{10b}$$

Now (10a) is just Bessel's equation of index $2l + 1$ whose general solution is the cylindrical functions $\mathcal{C}_{2l+1}(z)$, an arbitrary linear combination of the Bessel function $J_{2l+1}(z)$, and the Weber function, $Y_{2l+1}(z)$. These cylindrical functions obey the usual recursion formulas for Bessel functions:

$$\begin{aligned} \mathcal{C}_{n-1}(z) + \mathcal{C}_{n+1}(z) &= \frac{2n}{z}\,\mathcal{C}_n(z) \\ z\mathcal{C}_n'(z) + n\mathcal{C}_n(z) &= z\mathcal{C}_{n-1}(z). \end{aligned} \tag{11}$$

By utilizing these relations and the definition (8) of the Bessel operator ∇_l, it can be shown straightforwardly that

$$\nabla_l\left\{\frac{2l+2+q}{4(2+q)}\left(\frac{1}{2}z\right)^{q+2}\mathcal{C}_{2l+3+q} - \frac{1}{4(3+q)}\left(\frac{1}{2}z\right)^{q+3}\mathcal{C}_{2l+4+q}\right\} = \left(\frac{1}{2}z\right)^{q+4}\mathcal{C}_{2l+1+q}\,. \tag{12}$$

This equation permits solutions of (10b) to be generated directly. For set $q = 0$ in (12). Then this equation becomes identical with (10b) in the case $k = 1$, provided that $V_1^{(l)}(z)$ is defined by the bracket on the left side of (12), i.e. by

$$V_1^{(l)} = \frac{l+1}{4}\left(\frac{1}{2}z\right)^2\mathcal{C}_{2l+3} - \frac{1}{12}\left(\frac{1}{2}z\right)^3\mathcal{C}_{2l+4}\,. \tag{13}$$

Thus a solution has been found for the first of equations (10b).

For $k = 2$, the right hand side of (10b) is just $(\frac{1}{2}z)^4 V_1$ which, by (13), must contain two terms of the form $(\frac{1}{2}z)^{q+4}\mathcal{C}_{2l+1+q}$ with $q = 2$ and $q = 3$. These are both of the form found on the right of equation (12), so that $V_2^{(l)}(z)$ can be generated by two applications of the procedure outlined above. Higher terms are generated successively by the same process. For example, the first four coefficients of the series for $U^{(0,n)}(z)$ are

$$\begin{aligned} U_0^{(0)} &= \left(\frac{1}{2}z\right)\mathcal{C}_1 \\ U_1^{(0)} &= \frac{1}{4}\left(\frac{1}{2}z\right)^3\mathcal{C}_3 - \frac{1}{12}\left(\frac{1}{2}z\right)^4\mathcal{C}_4 \\ U_2^{(0)} &= \frac{1}{16}\left(\frac{1}{2}z\right)^5\mathcal{C}_5 - \frac{1}{30}\left(\frac{1}{2}z\right)^6\mathcal{C}_6 + \frac{1}{288}\left(\frac{1}{2}z\right)^7\mathcal{C}_7 \\ U_3^{(0)} &= \frac{1}{64}\left(\frac{1}{2}z\right)^7\mathcal{C}_7 - \frac{71}{6{,}720}\left(\frac{1}{2}z\right)^8\mathcal{C}_8 + \frac{11}{5{,}760}\left(\frac{1}{2}z\right)^9\mathcal{C}_9 - \frac{1}{10{,}368}\left(\frac{1}{2}z\right)^{10}\mathcal{C}_{10}\,. \end{aligned} \tag{14}$$

It has been assumed, and will be proven in the Appendix, that the series generated above and illustrated in (14) is itself analytic and uniformly convergent in z so that it may be differentiated term by term with respect to z to yield a series for the derivatives of the solutions of equation (3). Such a series is discussed further in section 5.

3. Some important particular solutions. The series generated above yields a general solution of equation (3) because the linear combination $\alpha J_m(z) + \beta Y_m(z)$ to be inserted for the cylindrical function $\mathcal{C}_m$ remains entirely arbitrary, so that the coefficients $U_k^{(l)}(z)$ are in fact ambiguously defined. For physical application of the series it is necessary to examine the effect of removing this ambiguity by particular choices of the constants

α and β. This examination is facilitated by defining two sets of coefficients, ${}^0U_k^{(l)}(z)$ and ${}^1U_k^{(l)}(z)$, which are gained from the ambiguous coefficients $U_k^{(l)}(z)$ by substituting $J_m(z)$ and $Y_m(z)$ respectively for the cylindrical functions $\mathfrak{C}_m(z)$. Two particular independent solutions of (3), ${}^0U^{(l,n)}(z)$ and ${}^1U^{(l,n)}(z)$, may then be defined as the result of applying the summation (8) to the coefficients ${}^0U_k^{(l)}(z)$ and ${}^1U_k^{(l)}(z)$, respectively.

The most general solution of (3) may now be written in the form

$$f(l, n)\,{}^0U^{(l,n)}(z) + g(l, n)\,{}^1U^{(l,n)}(z) \equiv \sum_{k=0}^{\infty} n^{-2k}[f(l, n)\,{}^0U_k^{(l)}(z) + g(l, n)\,{}^1U_k^{(l)}(z)], \tag{15}$$

where $f(l, n)$ and $g(l, n)$ are entirely arbitrary. Important particular solutions of (3) are gained by specifying these arbitrary functions in (15).

Since the Bessel functions all have zeros and the Weber functions all have poles at the origin, it can be shown that ${}^0U^{(l,n)}(z)$ is the only particular solution of (3) with a zero at the origin. It must therefore be identical, except in amplitude, with the particular solution $M_{n,m}(y)$ of (1), and a comparison of the leading terms (in z) of the expansions of the two series yields

$$M_{n,l+1/2}(z^2/4n) = n^{-l-1}\Gamma(2l + 2)\,{}^0U^{(l,n)}(z). \tag{16}$$

A second solution of physical interest, the only particular solution which goes to zero as z goes to infinity, may be discovered by a comparison of the series developed above with a series solution given by Wannier. In the paper previously noted, Wannier defines two solutions of (5), $J_{2l+1}^n(z)$ and $N_{2l+1}^n(z)$, by the formulas

$$\begin{aligned} J_{2l+1}^n(z) &\equiv \frac{n^{l+1}}{(1/2\,z)\Gamma(2l + 2)}\,M_{n,l+1/2}(z^2/4n) \\ N_{2l+1}^n(z) &\equiv \frac{1}{\sin(2l + 1)\pi}\left[\frac{\Gamma(n + l + 1)}{\Gamma(n - l)n^{2l+1}}\,J_{2l+1}^n(z)\cos(2l + 1)\pi - J_{-2l-1}^n(z)\right]. \end{aligned} \tag{17}$$

These solutions are shown to be independent and well defined for all values of l and n. Wannier further proves that the particular solution which goes to zero at infinity may be written[5]

$$\begin{aligned} W_{n,l+1/2}(z^2/4n) = (z^2/4n)^{1/2}[&\Gamma(n + l + 1)n^{-l-1/2}J_{2l+1}^n(z)\cos(n - l - 1)\pi \\ &+ \Gamma(n - l)n^{l+1/2}N_{2l+1}^n(z)\sin(n - l - 1)\pi]. \end{aligned} \tag{18}$$

Since the series expansion of $M_{n,l+1/2}$ is well known, equations (17) and (18) completely determine the expansion of $W_{n,l+1/2}$ for any value of l. It follows that if, for a given fixed value of l, the first $m + 1$ coefficients ${}^0U_k^{(l)}$ and ${}^1U_k^{(l)}$ have been developed by the generating procedure, the functions $f(l, n)$ and $g(l, n)$ may be determined (to terms in n^{-2m}) by explicit comparison of the series expansions (in z) of (15) and (18). If $2l + 1$ is not an integer, it is convenient to compare the coefficients of the terms in $(\frac{1}{2}z)^{-2l}$ and in $(\frac{1}{2}z)^{2l+2}$ in the two series; if $2l + 1$ is an integer the coefficient of the terms $(\frac{1}{2}z)^{2l+2}$ and $(\frac{1}{2}z)^{2l+2}\log(\frac{1}{2}z)$ are most conveniently compared. In the latter case, $2l + 1$ an integer, these two terms of (18) are given by

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[5]Wannier's paper has $-\Gamma(n - l)n^{l+1/2}\cdots$, a discrepancy which I assume to be due to a misprint in the original.

$$W_{n,l+1/2}(z^2/4n) = \cdots + \frac{\Gamma(n+l+1)}{n^{l+1}} \frac{(1/2\, z)^{2l+2}}{(2l+1)!} \Big\{ \cos(n-l-1)\pi$$

$$+ \frac{1}{\pi} \sin(n-l-1)\pi \Big[2 \log\left(\frac{1}{2} z\right) + 2\gamma - \sum_{m=1}^{2l+1} \frac{1}{m} + \Psi(n-l) - \log(n) \tag{19}$$

$$+ \Gamma(n-l)(2l+1)! \sum_{r=0}^{2l} (-1)^r \frac{(2l-r)!}{2^{2l+1-r}(2l+1-r)!\Gamma(n+l+1-r)} \Big] \Big\} + \cdots,$$

in which $\Psi(x) = d/dx \log \Gamma(x)$ and $\gamma =$ Euler's constant.

This procedure has been carried out for the two most important cases, $l = 0$ and $l = 1$. In both cases the manipulation yields (to terms in n^{-10})

$$\begin{aligned} W_{n,l+1/2}(z^2/4n) = n^{-l-1}\Gamma(n+l+1)[&\cos(n-l-1)\pi\, {}^0U^{(l,n)}(z) \\ &+ \sin(n-l-1)\pi\, {}^1U^{(l,n)}(z)]. \end{aligned} \tag{20}$$

There are additional theoretical reasons for supposing the equation (20) is in fact valid for all values of l, integral and non-integral, but a general proof of this result has not yet been given. Until such a proof is produced the method outlined above may be used to produce an equivalent result for any value of l for which the coefficients ${}^0U_k^{(l)}$ and ${}^1U_k^{(l)}$ have been developed.

4. An alternate generating procedure. Since the Weber functions have not been tabulated for large indices, it is convenient to develop the formulas for $U_k^{(l)}(z)$ so that they involve only $\mathcal{C}_0(z)$ and $\mathcal{C}_1(z)$. This may be accomplished by repeated application of the first of the recursion formulas (11) to the functions $V_k^{(l)}(z)$ generated by the method of section 2, but this reduction is arduous and may conveniently be replaced by the procedure sketched below.

The function $V_0^{(l)}(z)$ is just $\mathcal{C}_{2l+1}(z)$, and this may, by application of the recursion formulas, be rewritten in the form

$$V_0^{(l)}(z) = \sum_{i=m}^{M} a_i \left(\frac{1}{2} z\right)^{2i} \mathcal{C}_0 + \sum_{i=n}^{N} b_i \left(\frac{1}{2} z\right)^{2i+1} \mathcal{C}_1, \tag{21}$$

where the constants a_i and b_i are known rational numbers and the constants m, n, M, and N are known integers.

The function $V_1^{(l)}(z)$ must be expressible in the form

$$V_1^{(l)}(z) = \sum_{i=m}^{M+2} \alpha_i \left(\frac{1}{2} z\right)^{2i} \mathcal{C}_0 + \sum_{i=n}^{N+2} \beta_i \left(\frac{1}{2} z\right)^{2i+1} \mathcal{C}_1, \tag{22}$$

where the constants α_i and β_i are unknown rational numbers which can be determined by applying the differential equation (10b) to (21) and (22). This application of the differential equation is facilitated by the use of the equations

$$\begin{aligned} \nabla_l\left[\left(\frac{1}{2} z\right)^q \mathcal{C}_0\right] &= -4q\left(\frac{1}{2} z\right)^{q+1} \mathcal{C}_1 + [q^2 - (2l+1)^2]\left(\frac{1}{2} z\right)^q \mathcal{C}_0 \\ \nabla_l\left[\left(\frac{1}{2} z\right)^q \mathcal{C}_1\right] &= 4q\left(\frac{1}{2} z\right)^{q+1} \mathcal{C}_0 + [(q-1)^2 - (2l+1)^2]\left(\frac{1}{2} z\right)^q \mathcal{C}_1, \end{aligned} \tag{23}$$

which are immediate consequences of (6) and (11). This procedure reduces $V_1^{(l)}(z)$ to the form (21) after which the method may be reapplied to the generation of $V_2^{(l)}(z)$, etc.

It must, however, be noted that this procedure, in contrast to that described in section 2, does not always uniquely determine the functions $V_k^{(l)}(z)$, for the differential equation (10b) is normally satisfied by functions of the form (21) and (22) for all values of certain of the coefficients α_i and β_i. This will be understood when it is observed that if, with $V_{k-1}^{(l)}(z)$ given, a function $V_k^{(l)}(z)$ is found to satisfy (10b), then the new function $V_k^{(l)}(z) + \alpha V_0^{(l)}(z)$ will also satisfy (10b) for any α, a fact which follows directly from (10a). This ambiguity does not affect the legitimacy of the generating procedure, for although the quantity $\alpha V_0^{(l)}(z)$ may be added to any $V_k^{(l)}(z)$, the quantity thus added will, by (10b), affect the formulas for $V_{k+1}^{(l)}(z)$, $V_{k+2}^{(l)}(z)$, etc. It is in fact readily seen from (10b) that the net effect of adding $\alpha V_0^{(l)}(z)$ to the coefficient $V_k^{(l)}(z)$ is just to increase the amplitude of the sum of the series, i.e. of $V^{(l,n)}(z)$, by the factor $(1 + \alpha/n^{2k})$.

This new generating procedure was used in the preparation of the tables which follow. For simplicity of computation all those coefficients, α_i and β_i, which were not explicitly determined by the procedure were set equal to zero, thus reducing the complexity of the formulas for the coefficients $V_k^{(l)}(z)$. It is readily seen that, for $l = 0$, the tabulated coefficients are just those which would have been gained using the original generating procedure. For $l = 1$ the tabulated coefficients differ from those provided by the original procedure, but in this case the sums of the series gained with the tabulated coefficients may be made equal to the sums of the series gained with the standard coefficients described in section 2 by multiplying the former with the amplitude factor $n^2/(n^2 - 1)$. This modification of the amplitudes of the solutions is of no significance except when it is necessary to use (16) and (20) for explicit computation of Whittaker's functions, $M_{n,m}$ and $W_{n,m}$.

5. The tables. Table I below lists the formulas for $U_k^{(0)}(z)$ with $k = 0$ through 7. Table II gives a similar list for the functions $U_k^{(1)}(z)$. Tables III and IV list corresponding formulas for the functions $D_k^{(0)}(z)$ and $D_k^{(1)}(z)$ which are defined by

$$D_k^{(l)}(z) \equiv \left(\frac{1}{2} z\right) \frac{d}{dz} U_k^{(l)}(z), \tag{24}$$

from which the functions

$$D^{(l,n)}(z) \equiv \sum_{k=0}^{\infty} n^{-2k} D_k^{(l)}(z) = \frac{d}{dz} U^{(l,n)}(z) \tag{25}$$

may be computed.

Tables V through VIII list values of the functions ${}^0U_k^{(l)}(z)$, ${}^1U_k^{(l)}(z)$, ${}^0D_k^{(l)}(z)$, and ${}^1D_k^{(l)}(z)$ for $l = 0$ and 1, $k = 0$ through 7, and $z = 3.5(0.5)7.5$. As before a superscript zero preceding the function indicates substitution of a Bessel function for the corresponding cylindrical function, and a superscript one connotes use of the Weber function. Persons interested in utilizing these numerical results may also find useful the *Tables of Coulomb Wave Functions* recently prepared by the National Applied Mathematics Laboratory of the National Bureau of Standards. These supply a single irregular solution of the wave equation for $l = 0$ and for values of z smaller than those listed in our tables.

6. Acknowledgments. The problem treated in this paper arose during the preparation of a doctoral dissertation under the direction of Professor J. H. VanVleck. It is a privilege to acknowledge my gratitude for his constant encouragement and advice. I am also indebted to Dr. Robert Jastrow for interesting discussions of the problem.

Since completing this paper, my attention has been called to the fact that the generating procedure developed in section 2 above has been previously discussed by Yost, Wheeler, and Breit[6] in treating the formally identical problem of the wave equation with a repulsive Coulomb field (proton-proton scattering). The additional analytic and numerical material displayed here seems more than sufficient to justify the presentation of this independent investigation.

APPENDIX

The existence of an expansion in $1/n^2$. We may seek a solution of equation (5) in the form of a power series in z,

$$V^{(l,n)}(z) = z^{\alpha} \sum_{k=0}^{\infty} c_k z^k \tag{26}$$

Throughout its circle of convergence this series may be inserted in the differential equation, and manipulations identical with those used in producing solutions of Bessel's equation show that the two series

$$V^{(l,n)}(z) = \left(\frac{1}{2}z\right)^{\mp(2l+1)} \sum_{k=0}^{\infty} (-1)^k a_k \left(\frac{1}{2}z\right)^{2k} \tag{27}$$

are solutions of (5) for an arbitrary value of a_0, providing that the coefficients a_k are generated by the formulas

$$a_1 = a_0[1 \pm (2l + 1)]^{-1} \tag{28a}$$

$$a_k = \frac{a_{k-1} + (a_{k-2}/4n^2)}{k[k \pm (2l + 1)]} \tag{28b}$$

In equations (28a) and (28b) the positive or the negative signs are to be taken together, so that, if $2l + 1$ is not an integer, these equations define two independent solutions of (5). That choice of sign which makes $\pm(2l + 1) > 0$ produces a solution which is regular at $z = 0$; the other choice of sign yields a solution which is irregular there. These two solutions will hereafter be distinguished as the "regular" and the "irregular" series, respectively.

When $2l + 1$ is an integer, equations (27) and (28) define only the regular solution of the equation, but in this event a second irregular solution may be defined by any of the usual[7] devices developed for Bessel functions. In this paper we make the expedient assumption that $2l + 1$ is not an integer and then produce two independent solutions of (5), ${}^0V^{(l,n)}(z)$ and ${}^1V^{(l,n)}(z)$, which remain finite and independent as $2l + 1$ is varied continuously through any integer or zero. Because of their continuity as functions of l these solutions may be assumed valid for $2l + 1$ an integer, and this assumption may be rigorously justified by standard methods.

We now investigate the convergence of the two series subject to the simplifying restriction that n and l be real variables, z remaining complex. It then appears from (28b) that all the coefficients a_k of the regular series have the same sign as a_0, so it follows directly that

[6]F. L. Yost, J. A. Wheeler, and G. Breit, Phys. Rev. **49**, p. 174 (1936).

[7]G. N. Watson, *Theory of Bessel functions*, (Cambridge University Press, 1945), second edition, §§3.5 ff.

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)[k+1 \pm (2l+1)]} \left\{ 1 + \frac{k[k \pm (2l+1)]}{4n^2 + (a_{k-2}/a_{k-1})} \right\}$$
$$< \frac{1}{(k+1)[k+1 \pm (2l+1)]} + \frac{1}{4n^2}. \tag{29}$$

The inequality in (29) provides a very weak condition on a_{k+1}/a_k, because it is gained by dropping the entire positive term $a_{k-2}/4n^2a_{k-1}$ from the positive denominator $1 + a_{k-2}/4n^2a_{k-1}$ in the equality. Stronger conditions may be gained by successive application of (29) to itself, for (29) may be used to set a lower bound on a_{k-2}/a_{k-1}, and this bound may be used to write a second and stronger form of (29). The process may be repeated indefinitely, and the p-th form of (29) gained in this manner is

$$\frac{a_{k+1}}{a_k} < \frac{(p+2)}{2(k+1-2p)[k+1-2p \pm (2l+1)]} + \frac{1}{(p+1)4n^2} \tag{30}$$

a formula which reduces to (29) when $p = 0$ and which is valid for all $k > 2p - 1 \mp (2l+1)$.

It follows that for all $|n| \geq n_0$ and for all $|z| < R$ (where n_0 and R are arbitrary positive numbers) the regular series in (27) converges absolutely and uniformly, for if p in (30) is chosen to be the largest integer less than $R^2/16n_0^2$, an integer k_0 may always be found such that $a_{k+1} | \frac{1}{2}z |^2/a_k < 1$ for all $k > k_0$. It follows that the regular series defined by (27) converges to an analytic function of z in any bounded domain in the z plane and that this function is uniformly continuous in n for all real n such that $|n| \geq n_0 > 0$.

It may next be noted that equation (28) generates coefficients a_k which are given by finite polynomials of the form

$$a_k = \sum_{m=0}^{M_k} b_m^k n^{-2m} \tag{31}$$

where $M_k = \frac{1}{2}k$ or $\frac{1}{2}(k-1)$, whichever is an integer. Further, if a_k is a coefficient of the regular series, the polynomial coefficients b_m^k must be positive quantities.

By considering (27) and (31), we may now define the "complete series" for $V^{(l,n)}(z)$ as the series in which each $b_m^k\, n^{-2m}\, (\frac{1}{2}z)^{2k \mp (2l+1)}$ is considered a separate term and in which for each value of k, the summation is carried out over values of m from 0 to M_k before k is increased by unity. It then follows from the absolute convergence of (27) and from the uniformly positive values of the b_m^k that the "complete series" for the regular solution also converges absolutely and uniformly, so that the terms of the "complete" series may be rearranged to provide an analytic solution of (5) in the form

$$V^{(l,n)}(z) = \sum_{k=0}^{\infty} V_k^{(l)}(z) n^{-2k} \tag{7}$$

The functions $V_k^{(l)}(z)$ are defined by uniformly and absolutely convergent power series in z, so that they are analytic functions of z, and the entire series (7) converges uniformly whenever $|n| \geq n_0$.

The existence of the series (7) has so far been proven only in the case of the regular solution of (5), but the proof is readily extended to the irregular solution and hence to

the general solution, which is an arbitrary linear combination of the two. For consider a series (27) in which the coefficients a_k are replaced by coefficients a'_k determined by

$$a'_1 = a'_0 \mid 1 \pm (2l + 1) \mid^{-1} \tag{32a}$$

$$a'_k = \frac{a'_{k-1} + (a'_0/4n^2)}{k \mid k \pm (2l + 1) \mid} \tag{32b}$$

This series contains only positive terms, so that the entire proof applied to the regular series holds for it. Further, if the choice of sign which produced the irregular series in (27) is utilized in (32), then every a'_k and every b'^k_m is greater than or equal to the corresponding a_k or b^k_m of the irregular series. It follows from the comparison test that the "complete series" for the irregular solution must converge absolutely and uniformly, so that the rearrangement of terms is again permissible. This is sufficient to justify entirely the assumption made in section 2, above.

TABLE I. Formulas for the coefficients $U_k^{(0)}(z)$ with $k = 0$ through 7.

$$U_0^{(0)} = \left(\frac{z}{2}\right)\mathfrak{C}_1$$

$$U_1^{(0)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4\mathfrak{C}_0 + \frac{1}{12}\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$U_2^{(0)} = \left[\frac{1}{120} - \frac{1}{120}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^4\mathfrak{C}_0 - \left[\frac{1}{120} - \frac{1}{80}\left(\frac{z}{2}\right)^2 + \frac{1}{288}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$U_3^{(0)} = -\left[\frac{1}{252} - \frac{1}{252}\left(\frac{z}{2}\right)^2 + \frac{79}{60{,}480}\left(\frac{z}{2}\right)^4 - \frac{1}{10{,}368}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^4\mathfrak{C}_0 + \left[\frac{1}{252} - \frac{1}{168}\left(\frac{z}{2}\right)^2\right.$$
$$\left. + \frac{179}{60{,}480}\left(\frac{z}{2}\right)^4 - \frac{13}{25{,}920}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$U_4^{(0)} = \left[\frac{1}{240} - \frac{1}{240}\left(\frac{z}{2}\right)^2 + \frac{19}{12{,}096}\left(\frac{z}{2}\right)^4 - \frac{97}{362{,}880}\left(\frac{z}{2}\right)^6 + \frac{7}{414{,}720}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^4\mathfrak{C}_0$$
$$- \left[\frac{1}{240} - \frac{1}{160}\left(\frac{z}{2}\right)^2 + \frac{5}{1{,}512}\left(\frac{z}{2}\right)^4 - \frac{115}{145{,}152}\left(\frac{z}{2}\right)^6 + \frac{403}{4{,}838{,}400}\left(\frac{z}{2}\right)^8\right.$$
$$\left. - \frac{1}{497{,}664}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$U_5^{(0)} = -\left[\frac{1}{132} - \frac{1}{132}\left(\frac{z}{2}\right)^2 + \frac{21}{7{,}040}\left(\frac{z}{2}\right)^4 - \frac{5}{8{,}448}\left(\frac{z}{2}\right)^6 + \frac{1{,}977}{31{,}933{,}440}\left(\frac{z}{2}\right)^8\right.$$
$$\left. - \frac{131}{43{,}545{,}600}\left(\frac{z}{2}\right)^{10} + \frac{1}{29{,}859{,}840}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^4\mathfrak{C}_0$$
$$+ \left[\frac{1}{132} - \frac{1}{88}\left(\frac{z}{2}\right)^2 + \frac{389}{63{,}360}\left(\frac{z}{2}\right)^4 - \frac{17}{10{,}560}\left(\frac{z}{2}\right)^6\right.$$
$$\left. + \frac{479}{2{,}128{,}896}\left(\frac{z}{2}\right)^8 - \frac{701}{43{,}545{,}600}\left(\frac{z}{2}\right)^{10} + \frac{13}{29{,}859{,}840}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$U_6^{(0)} = \left[\frac{691}{32,760} - \frac{691}{32,760}\left(\frac{z}{2}\right)^2 + \frac{229}{27,027}\left(\frac{z}{2}\right)^4 - \frac{3,313}{1,853,280}\left(\frac{z}{2}\right)^6 + \frac{43,037}{197,683,200}\left(\frac{z}{2}\right)^8 - \frac{160,361}{10,378,368,000}\left(\frac{z}{2}\right)^{10} + \frac{509}{870,912,000}\left(\frac{z}{2}\right)^{12} - \frac{1}{119,439,360}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0$$
$$- \left[\frac{691}{32,760} - \frac{691}{21,840}\left(\frac{z}{2}\right)^2 + \frac{14,929}{864,864}\left(\frac{z}{2}\right)^4 - \frac{6,977}{1,482,624}\left(\frac{z}{2}\right)^6 + \frac{47,951}{65,894,400}\left(\frac{z}{2}\right)^8 - \frac{393,847}{5,930,496,000}\left(\frac{z}{2}\right)^{10} + \frac{461,819}{134,120,448,000}\left(\frac{z}{2}\right)^{12} - \frac{59}{696,729,600}\left(\frac{z}{2}\right)^{14} + \frac{1}{2,149,908,480}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$U_7^{(0)} = -\left[\frac{273}{3,276} - \frac{273}{3,276}\left(\frac{z}{2}\right)^2 + \frac{26,609}{786,240}\left(\frac{z}{2}\right)^4 - \frac{6,953}{943,488}\left(\frac{z}{2}\right)^6 + \frac{3,569}{3,706,560}\left(\frac{z}{2}\right)^8 - \frac{11,717}{148,262,400}\left(\frac{z}{2}\right)^{10} + \frac{404,561}{99,632,332,800}\left(\frac{z}{2}\right)^{12} - \frac{65,539}{536,481,792,000}\left(\frac{z}{2}\right)^{14} + \frac{11}{6,270,566,400}\left(\frac{z}{2}\right)^{16} - \frac{1}{180,592,312,320}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0$$
$$+ \left[\frac{273}{3,276} - \frac{273}{2,184}\left(\frac{z}{2}\right)^2 + \frac{53,909}{786,240}\left(\frac{z}{2}\right)^4 - \frac{45,011}{2,358,720}\left(\frac{z}{2}\right)^6 + \frac{5,363}{1,729,728}\left(\frac{z}{2}\right)^8 - \frac{279,187}{889,574,400}\left(\frac{z}{2}\right)^{10} + \frac{286,511}{14,233,190,400}\left(\frac{z}{2}\right)^{12} - \frac{919,637}{1,162,377,216,000}\left(\frac{z}{2}\right)^{14} + \frac{353}{20,901,888,000}\left(\frac{z}{2}\right)^{16} - \frac{61}{451,480,780,800}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

TABLE II. Formulas for the coefficients $U_k^{(1)}(z)$ with $k = 0$ through 7.

$$U_0^{(1)} = -2\mathfrak{C}_0 + \left[2 - \left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^{-1} \mathfrak{C}_1$$

$$U_1^{(1)} = \frac{1}{12}\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \frac{1}{4}\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$U_2^{(1)} = \frac{11}{720}\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 - \left[\frac{11}{720} - \frac{1}{288}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

$$U_3^{(1)} = -\left[\frac{31}{15,120} - \frac{31}{20,160}\left(\frac{z}{2}\right)^2 + \frac{1}{10,368}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 + \left[\frac{31}{15,120} - \frac{31}{12,096}\left(\frac{z}{2}\right)^2 + \frac{1}{1,440}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

$$U_4^{(1)} = \left[\frac{41}{30,240} - \frac{41}{40,320}\left(\frac{z}{2}\right)^2 + \frac{481}{1,814,400}\left(\frac{z}{2}\right)^4 - \frac{13}{622,080}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 - \left[\frac{41}{30,240} - \frac{41}{24,192}\left(\frac{z}{2}\right)^2 + \frac{799}{1,209,600}\left(\frac{z}{2}\right)^4 - \frac{2,111}{21,772,800}\left(\frac{z}{2}\right)^6 + \frac{1}{497,664}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

$$U_5^{(1)} = -\left[\frac{31}{15,840} - \frac{31}{21,120}\left(\frac{z}{2}\right)^2 + \frac{25}{59,136}\left(\frac{z}{2}\right)^4 - \frac{5,557}{95,800,320}\left(\frac{z}{2}\right)^6 + \frac{11}{3,225,600}\left(\frac{z}{2}\right)^8 - \frac{1}{29,859,840}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 + \left[\frac{31}{15,840} - \frac{31}{12,672}\left(\frac{z}{2}\right)^2 + \frac{661}{665,280}\left(\frac{z}{2}\right)^4 - \frac{3,599}{19,160,064}\left(\frac{z}{2}\right)^6 + \frac{1,471}{87,091,200}\left(\frac{z}{2}\right)^8 - \frac{1}{1,990,656}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

$$U_6^{(1)} = \left[\frac{10,331}{2,162,160} - \frac{10,331}{2,882,880}\left(\frac{z}{2}\right)^2 + \frac{79,021}{74,131,200}\left(\frac{z}{2}\right)^4 - \frac{68,543}{415,134,720}\left(\frac{z}{2}\right)^6 + \frac{24,931}{1,761,177,600}\left(\frac{z}{2}\right)^8 - \frac{3,239}{5,225,472,000}\left(\frac{z}{2}\right)^{10} + \frac{1}{107,495,424}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 - \left[\frac{10,331}{2,162,160} - \frac{10,331}{1,729,728}\left(\frac{z}{2}\right)^2 + \frac{2,251}{915,200}\left(\frac{z}{2}\right)^4 - \frac{147,967}{296,524,800}\left(\frac{z}{2}\right)^6 + \frac{650,857}{11,623,772,160}\left(\frac{z}{2}\right)^8 - \frac{229,793}{67,060,224,000}\left(\frac{z}{2}\right)^{10} + \frac{221}{2,351,462,400}\left(\frac{z}{2}\right)^{12} - \frac{1}{2,149,908,480}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

$$U_7^{(1)} = -\left[\frac{3,421}{196,560} - \frac{3,421}{262,080}\left(\frac{z}{2}\right)^2 + \frac{40,907}{10,378,368}\left(\frac{z}{2}\right)^4 - \frac{57,131}{88,957,440}\left(\frac{z}{2}\right)^6 + \frac{2,743}{43,929,600}\left(\frac{z}{2}\right)^8 - \frac{1,837,343}{498,161,664,000}\left(\frac{z}{2}\right)^{10} + \frac{200,159}{1,609,445,376,000}\left(\frac{z}{2}\right)^{12} - \frac{1}{522,547,200}\left(\frac{z}{2}\right)^{14} + \frac{1}{180,592,312,320}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^6 \mathfrak{C}_0 + \left[\frac{3,421}{196,560} - \frac{3,421}{157,248}\left(\frac{z^2}{2}\right) + \frac{9,749}{1,081,080}\left(\frac{z}{2}\right)^4 - \frac{235,111}{124,540,416}\left(\frac{z}{2}\right)^6 + \frac{819,631}{3,558,297,600}\left(\frac{z}{2}\right)^8 - \frac{407,413}{23,721,984,000}\left(\frac{z}{2}\right)^{10} + \frac{800,593}{1,046,139,494,400}\left(\frac{z}{2}\right)^{12} - \frac{751}{41,803,776,000}\left(\frac{z}{2}\right)^{14} + \frac{11}{75,246,796,800}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^5 \mathfrak{C}_1$$

TABLE III. Formulas for the coefficients $D_k^{(0)}(z)$ with $k = 0$ through 7.

$$D_0^{(0)} = \left(\frac{z}{2}\right)^2 \mathfrak{C}_0$$

$$D_1^{(0)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 + \left[\frac{1}{12} + \frac{1}{12}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$D_2^{(0)} = \left[\frac{1}{120} - \frac{1}{80}\left(\frac{z}{2}\right)^2 - \frac{1}{288}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \left[\frac{1}{120} - \frac{1}{60}\left(\frac{z}{2}\right)^2 + \frac{1}{480}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1$$

$$
\begin{aligned}
D_3^{(0)} = {} & -\left[\frac{1}{252} - \frac{1}{168}\left(\frac{z}{2}\right)^2 + \frac{137}{60,480}\left(\frac{z}{2}\right)^4 + \frac{1}{51,840}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 \\
& + \left[\frac{1}{252} - \frac{1}{126}\left(\frac{z}{2}\right)^2 + \frac{11}{2,240}\left(\frac{z}{2}\right)^4 - \frac{127}{181,440}\left(\frac{z}{2}\right)^6 - \frac{1}{10,368}\left(\frac{z}{2}\right)^8\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1
\end{aligned}
$$

$$
\begin{aligned}
D_4^{(0)} = {} & \left[\frac{1}{240} - \frac{1}{160}\left(\frac{z}{2}\right)^2 + \frac{1}{336}\left(\frac{z}{2}\right)^4 - \frac{79}{145,152}\left(\frac{z}{2}\right)^6 + \frac{87}{4,838,400}\left(\frac{z}{2}\right)^8\right. \\
& \left. + \frac{1}{497,664}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \left[\frac{1}{240} - \frac{1}{120}\left(\frac{z}{2}\right)^2 + \frac{29}{5,040}\left(\frac{z}{2}\right)^4 - \frac{29}{18,144}\left(\frac{z}{2}\right)^6\right. \\
& \left. + \frac{433}{2,903,040}\left(\frac{z}{2}\right)^8 + \frac{1}{207,360}\left(\frac{z}{2}\right)^{10}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1
\end{aligned}
$$

$$
\begin{aligned}
D_5^{(0)} = {} & -\left[\frac{1}{132} - \frac{1}{88}\left(\frac{z}{2}\right)^2 + \frac{367}{63,360}\left(\frac{z}{2}\right)^4 - \frac{19}{14,080}\left(\frac{z}{2}\right)^6 + \frac{1,559}{10,644,480}\left(\frac{z}{2}\right)^8\right. \\
& \left. - \frac{1}{201,600}\left(\frac{z}{2}\right)^{10} - \frac{1}{5,971,968}\left(\frac{z}{2}\right)^{12}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 + \left[\frac{1}{132} - \frac{1}{66}\left(\frac{z}{2}\right)^2\right. \\
& + \frac{229}{21,120}\left(\frac{z}{2}\right)^4 - \frac{73}{21,120}\left(\frac{z}{2}\right)^6 + \frac{1,135}{2,128,896}\left(\frac{z}{2}\right)^8 - \frac{791}{22,809,600}\left(\frac{z}{2}\right)^{10} \\
& \left. + \frac{41}{1,045,094,400}\left(\frac{z}{2}\right)^{12} + \frac{1}{29,859,840}\left(\frac{z}{2}\right)^{14}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1
\end{aligned}
$$

$$
\begin{aligned}
D_6^{(0)} = {} & \left[\frac{691}{32,760} - \frac{691}{21,840}\left(\frac{z}{2}\right)^2 + \frac{14,383}{864,864}\left(\frac{z}{2}\right)^4 - \frac{6,275}{1,482,624}\left(\frac{z}{2}\right)^6 + \frac{38,123}{65,894,400}\left(\frac{z}{2}\right)^8\right. \\
& - \frac{247,597}{5,930,496,000}\left(\frac{z}{2}\right)^{10} + \frac{165,269}{134,120,448,000}\left(\frac{z}{2}\right)^{12} + \frac{13}{1,393,459,200}\left(\frac{z}{2}\right)^{14} \\
& \left. - \frac{1}{2,149,908,480}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 - \left[\frac{691}{32,760} - \frac{691}{16,380}\left(\frac{z}{2}\right)^2 + \frac{14,747}{480,480}\left(\frac{z}{2}\right)^4\right. \\
& - \frac{26,855}{2,594,592}\left(\frac{z}{2}\right)^6 + \frac{19,957}{10,782,720}\left(\frac{z}{2}\right)^8 - \frac{29,777}{164,736,000}\left(\frac{z}{2}\right)^{10} \\
& \left. + \frac{2,154,983}{249,080,832,000}\left(\frac{z}{2}\right)^{12} - \frac{1}{10,752,000}\left(\frac{z}{2}\right)^{14} - \frac{1}{238,878,200}\left(\frac{z}{2}\right)^{16}\right]\left(\frac{z}{2}\right)^3 \mathfrak{C}_1
\end{aligned}
$$

$$
\begin{aligned}
D_7^{(0)} = {} & -\left[\frac{273}{3,276} - \frac{273}{2,184}\left(\frac{z}{2}\right)^2 + \frac{17,509}{262,080}\left(\frac{z}{2}\right)^4 - \frac{83,803}{4,717,440}\left(\frac{z}{2}\right)^6 + \frac{7,717}{2,882,880}\left(\frac{z}{2}\right)^8\right. \\
& - \frac{1,489}{6,220,800}\left(\frac{z}{2}\right)^{10} + \frac{111,901}{9,057,484,800}\left(\frac{z}{2}\right)^{12} - \frac{716,747}{2,324,754,432,000}\left(\frac{z}{2}\right)^{14} \\
& \left. + \frac{41}{62,705,664,000}\left(\frac{z}{2}\right)^{16} + \frac{67}{902,961,561,600}\left(\frac{z}{2}\right)^{18}\right]\left(\frac{z}{2}\right)^4 \mathfrak{C}_0 \\
& + \left[\frac{273}{3,276} - \frac{273}{1,638}\left(\frac{z}{2}\right)^2 + \frac{32,069}{262,080}\left(\frac{z}{2}\right)^4 - \frac{100,217}{2,358,720}\left(\frac{z}{2}\right)^6\right. \\
& + \frac{84,407}{10,378,368}\left(\frac{z}{2}\right)^8 - \frac{136,427}{148,262,400}\left(\frac{z}{2}\right)^{10} + \frac{176,149}{2,846,638,080}\left(\frac{z}{2}\right)^{12}
\end{aligned}
$$

$$- \frac{7,911,653}{3,487,131,648,000}\left(\frac{z}{2}\right)^{14} + \frac{4,001}{134,120,448,000}\left(\frac{z}{2}\right)^{16}$$
$$+ \frac{13}{32,248,627,200}\left(\frac{z}{2}\right)^{18} - \frac{1}{180,592,312,320}\left(\frac{z}{2}\right)^{20}\Bigg]\left(\frac{z}{2}\right)^{3}\mathfrak{C}_1$$

TABLE IV. Formulas for the coefficients $D_k^{(1)}(z)$ with $k = 0$ through 7.

$$D_0^{(1)} = \left[2 - \left(\frac{z}{2}\right)^2\right]\mathfrak{C}_0 - \left[2 - 2\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^{-1}\mathfrak{C}_1$$

$$D_1^{(1)} = -\frac{1}{12}\left(\frac{z}{2}\right)^4\mathfrak{C}_0 - \left[\frac{1}{4} + \frac{1}{12}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^3\mathfrak{C}_1$$

$$D_2^{(1)} = \left[\frac{11}{360} + \frac{1}{288}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^6\mathfrak{C}_0 - \left[\frac{11}{360} + \frac{7}{1,440}\left(\frac{z}{2}\right)^2\right]\left(\frac{z}{2}\right)^5\mathfrak{C}_1$$

$$D_3^{(1)} = -\left[\frac{31}{7,560} - \frac{31}{8,640}\left(\frac{z}{2}\right)^2 - \frac{11}{51,840}\left(\frac{z}{2}\right)^4\right]\left(\frac{z}{2}\right)^6\mathfrak{C}_0$$
$$+ \left[\frac{31}{7,560} - \frac{341}{60,480}\left(\frac{z}{2}\right)^2 + \frac{5}{4,032}\left(\frac{z}{2}\right)^4 + \frac{1}{10,368}\left(\frac{z}{2}\right)^6\right]\left(\frac{z}{2}\right)^5\mathfrak{C}_1$$

$$D_4^{(1)} = \Bigg[\frac{41}{15,120} - \frac{41}{17,280}\left(\frac{z}{2}\right)^2 + \frac{2,413}{3,628,800}\left(\frac{z}{2}\right)^4 - \frac{619}{21,772,800}\left(\frac{z}{2}\right)^6$$
$$- \frac{1}{497,664}\left(\frac{z}{2}\right)^8\Bigg]\left(\frac{z}{2}\right)^6\mathfrak{C}_0 - \Bigg[\frac{41}{15,120} - \frac{451}{120,960}\left(\frac{z}{2}\right)^2 + \frac{983}{604,800}\left(\frac{z}{2}\right)^4$$
$$- \frac{4,783}{21,772,800}\left(\frac{z}{2}\right)^6 - \frac{11}{1,244,160}\left(\frac{z}{2}\right)^8\Bigg]\left(\frac{z}{2}\right)^5\mathfrak{C}_1$$

$$D_5^{(1)} = -\Bigg[\frac{31}{7,920} - \frac{217}{63,360}\left(\frac{z}{2}\right)^2 + \frac{271}{241,920}\left(\frac{z}{2}\right)^4 - \frac{15,347}{95,800,320}\left(\frac{z}{2}\right)^6 + \frac{608}{87,091,200}\left(\frac{z}{2}\right)^8$$
$$+ \frac{7}{29,859,840}\left(\frac{z}{2}\right)^{10}\Bigg]\left(\frac{z}{2}\right)^6\mathfrak{C}_0 + \Bigg[\frac{31}{7,920} - \frac{31}{5,760}\left(\frac{z}{2}\right)^2 + \frac{667}{266,112}\left(\frac{z}{2}\right)^4$$
$$- \frac{9,859}{19,160,064}\left(\frac{z}{2}\right)^6 + \frac{10,379}{239,500,800}\left(\frac{z}{2}\right)^8 - \frac{37}{348,364,800}\left(\frac{z}{2}\right)^{10}$$
$$- \frac{1}{29,859,840}\left(\frac{z}{2}\right)^{12}\Bigg]\left(\frac{z}{2}\right)^5\mathfrak{C}_1$$

$$D_6^{(1)} = \Bigg[\frac{10,331}{1,081,080} - \frac{10,331}{1,235,520}\left(\frac{z}{2}\right)^2 + \frac{106,387}{37,065,600}\left(\frac{z}{2}\right)^4 - \frac{1,020,521}{2,075,673,600}\left(\frac{z}{2}\right)^6$$
$$+ \frac{313,097}{7,264,857,600}\left(\frac{z}{2}\right)^8 - \frac{308,233}{201,180,672,000}\left(\frac{z}{2}\right)^{10} - \frac{193}{18,811,699,200}\left(\frac{z}{2}\right)^{12}$$
$$+ \frac{1}{2,149,908,480}\left(\frac{z}{2}\right)^{14}\Bigg]\left(\frac{z}{2}\right)^6\mathfrak{C}_0 - \Bigg[\frac{10,331}{1,081,080} - \frac{10,331}{786,240}\left(\frac{z}{2}\right)^2$$
$$+ \frac{45,079}{7,207,200}\left(\frac{z}{2}\right)^4 - \frac{423,751}{296,524,800}\left(\frac{z}{2}\right)^6 + \frac{992,969}{5,811,886,080}\left(\frac{z}{2}\right)^8$$

$$- \frac{4{,}285{,}159}{435{,}891{,}456{,}000}\left(\frac{z}{2}\right)^{10} + \frac{887}{6{,}718{,}464{,}000}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{11}{2{,}149{,}908{,}480}\left(\frac{z}{2}\right)^{14}\Bigg]\left(\frac{z}{2}\right)^{5}\mathcal{C}_1$$

$$D_7^{(1)} = -\Bigg[\frac{3{,}421}{98{,}280} - \frac{3{,}421}{112{,}320}\left(\frac{z}{2}\right)^{2} + \frac{42{,}671}{3{,}991{,}680}\left(\frac{z}{2}\right)^{4} - \frac{1{,}223{,}947}{622{,}702{,}080}\left(\frac{z}{2}\right)^{6}$$

$$+ \frac{14{,}713}{71{,}165{,}952}\left(\frac{z}{2}\right)^{8} - \frac{558{,}461}{45{,}287{,}424{,}000}\left(\frac{z}{2}\right)^{10} + \frac{7{,}406{,}743}{20{,}922{,}789{,}888{,}000}\left(\frac{z}{2}\right)^{12}$$

$$- \frac{7}{5{,}971{,}968{,}000}\left(\frac{z}{2}\right)^{14} - \frac{11}{128{,}994{,}508{,}800}\left(\frac{z}{2}\right)^{16}\Bigg]\left(\frac{z}{2}\right)^{6}\mathcal{C}_0$$

$$+ \Bigg[\frac{3{,}421}{98{,}280} - \frac{37{,}631}{786{,}240}\left(\frac{z}{2}\right)^{2} + \frac{39{,}815}{1{,}729{,}728}\left(\frac{z}{2}\right)^{4} - \frac{52{,}667}{9{,}580{,}032}\left(\frac{z}{2}\right)^{6}$$

$$+ \frac{1{,}316{,}273}{1{,}779{,}148{,}800}\left(\frac{z}{2}\right)^{8} - \frac{1{,}370{,}671}{23{,}721{,}984{,}000}\left(\frac{z}{2}\right)^{10} + \frac{25{,}463{,}237}{10{,}461{,}394{,}944{,}000}\left(\frac{z}{2}\right)^{12}$$

$$- \frac{961}{25{,}751{,}126{,}016}\left(\frac{z}{2}\right)^{14} - \frac{17}{37{,}623{,}398{,}400}\left(\frac{z}{2}\right)^{16}$$

$$+ \frac{1}{180{,}592{,}312{,}320}\left(\frac{z}{2}\right)^{18}\Bigg]\left(\frac{z}{2}\right)^{5}\mathcal{C}_1$$

TABLE V. Values of the coefficients $^0U_k^{(0)}(z)$ and $^1U_k^{(0)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.5$

z \ k	0	1	2	3	4	5	6	7
3.5	+ .2404	+ .3585	+ .0593	+ .0040	+ .0001		$^0U_k^{(0)}(z)$	
4.0	− .1321	+ .4855	+ .1662	+ .0204	+ .0013			
4.5	− .5199	+ .4653	+ .3677	+ .0803	+ .0088	+ .0006		
5.0	− .8189	+ .1516	+ .6405	+ .2512	+ .0451	+ .0048		
5.5	− .9390	− .5591	+ .8194	+ .6326	+ .1844	+ .0304	+ .0033	
6.0	− .8301	−1.6394	+ .5094	+1.2653	+ .6126	+ .1543	+ .0248	+ .0028
6.5	− .5000	−2.8582	− .9198	+1.8590	+1.6560	+ .6394	+ .1500	+ .0241
7.0	− .0016	−3.7693	−4.1461	+1.2775	+3.5502	+2.1826	+ .7462	+ .1703
7.5	+ .5072	−3.7948	−9.4364	−3.0898	+5.4044	+6.0816	+3.0829	+ .9871
3.5	+ .7178	+ .0355	− .0362	− .0043	− .0003		$^1U_k^{(0)}(z)$	
4.0	+ .7959	+ .2879	− .0374	− .0138	− .0020			
4.5	+ .6772	+ .7016	+ .0522	− .0286	− .0073	+ .0009		
5.0	+ .3697	+1.1968	+ .3751	− .0125	− .0200	− .0018		
5.5	− .0653	+1.5768	+1.1173	+ .1787	− .0276	− .0124	− .0067	
6.0	− .5250	+1.5515	+2.3930	+ .9167	+ .0838	− .0297	− .0128	+ .0091
6.5	− .8908	+ .8266	+4.0212	+2.8948	+ .7845	+ .0375	− .0299	− .0127
7.0	−1.0593	− .7569	+5.2477	+6.9946	+3.4453	+ .7529	+ .0207	− .0512
7.5	− .9717	−3.0720	+4.5695	+13.5834	+11.1065	+4.3933	+ .8567	− .0145

TABLE VI. Values of the coefficients ${}^0U_k^{(1)}(z)$ and ${}^1U_k^{(1)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.0$

$z \backslash k$	0	1	2	3	4	5	6	7
							${}^0U_k^{(1)}(z)$	
3.5	+.6768	−.4812	−.1773	−.0175	−.0008			
4.0	+.8603	−.3974	−.3854	−.0712	−.0060	−.0003		
4.5	+.9556	−.0266	−.6660	−.2273	−.0328	−.0027		
5.0	+.9121	+.7015	−.8679	−.5861	−.1402	−.0184	−.0015	
5.5	+.7043	+1.7426	−.6349	−1.2141	−.4851	−.0990	−.0126	
6.0	+.3443	+2.8845	+.6039	−1.9323	−1.3738	−.4327	−.0822	−.0106
6.5	−.1149	+3.7384	+3.4891	−1.8745	−3.1495	−1.5606	−.4354	−.2526
7.0	−.5864	+3.8027	+8.3606	+1.0917	−5.5007	−4.6443	−1.9128	−.5038
							${}^1U_k^{(1)}(z)$	
3.5	−.6271	−.4019	+.0517	+.0143	+.0024			
4.0	−.3640	−.8184	−.0342	+.0343	+.0067	−.0019		
4.5	−.0203	−1.2730	−.3460	+.0369	+.0196	+.0006		
5.0	+.3657	−1.5819	−1.0580	−.0954	+.0392	+.0102	+.0052	
5.5	+.7270	−1.4944	−2.2842	−.6801	−.0028	+.0331	+.0081	
6.0	+.9847	−.7640	−3.8890	−2.3339	−.4337	+.0339	+.0256	+.0134
6.5	+1.0686	+.7416	−5.2455	−5.9138	−2.2843	−.2992	+.0439	+.1615
7.0	+.9383	+2.9197	−5.0617	−12.0115	−8.0027	−2.3942	−.2510	+.0885

TABLE VII. Values of the coefficients ${}^0D_k^{(0)}(z)$ and ${}^1D_k^{(0)}(z)$ with $k = 0$ through 7 and with $z = 3.5(0.5)7.5$

$z \backslash k$	0	1	2	3	4	5	6	7
							${}^0D_k^{(0)}(z)$	
3.5	−1.1641	+.5464	+.2399	+.0248	+.0012			
4.0	−1.5886	+.3094	+.6046	+.1219	+.0108	+.0005		
4.5	−1.6227	−.6451	+1.1299	+.4524	+.0696	+.0059		
5.0	−1.1100	−2.5143	+1.3511	+1.2957	+.3423	+.0473	+.0041	
5.5	−.0518	−5.0341	+.1217	+2.8315	+1.3213	+.2896	+.0387	
6.0	+1.3558	−7.2422	−4.5006	+4.2666	+4.0382	+1.4073	+.2830	+.0381
6.5	+2.7472	−7.5067	−14.4887	+1.9408	+9.5432	+5.5044	+1.6547	+.3222
7.0	+3.6760	−3.9742	−29.9598	−13.2338	+15.4978	+17.2609	+7.8914	+2.2065
7.5	+3.7454	+4.5633	−46.3111	−56.8456	+5.9852	41.6448	+30.6623	+12.3593
							${}^1D_k^{(0)}(z)$	
3.5	+.5789	+.5965	−.0599	−.0197	−.0035			
4.0	−.0678	+1.3490	+.1059	−.0547	−.0115	+.0030		
4.5	−.9857	+2.1480	+.7959	−.0496	−.0386	−.0030		
5.0	−1.9282	+2.4001	+2.5091	+.3255	−.0799	−.0280	−.0042	
5.5	−2.5673	+1.2654	+5.5298	+2.0747	+.0874	−.0904	−.0119	
6.0	−2.5938	−1.9924	+9.1250	+7.2941	+1.6581	−.0485	−.0705	−.0921
6.5	−1.8299	−7.4553	+10.4872	+18.8387	+8.7051	+1.4192	−.1036	−.1935
7.0	−.3179	−14.0041	+4.1085	+37.7230	+31.3594	+10.8055	+1.4318	−.2809
7.5	+1.6497	−19.0857	−17.2733	+56.5553	+87.7617	+51.1071	+14.8618	+1.7063

TABLE VIII. Values of the coefficients $^0D_k^{(1)}(z)$ and $^1D_k^{(1)}(z)$ with k = o through 7 and with $z = 3.5(0.5)7.0$

z \ k	0	1	2	3	4	5	6	7
							$^0D_k^{(1)}(z)$	
3.5	+ .7277	− .0749	− .5522	− .0941	− .0063			
4.0	+ .5962	+ .8377	−1.0240	− .3628	− .0441	− .0031		
4.5	+ .1473	+2.4529	−1.2668	−1.0756	− .2312	− .0281		
5.0	− .6210	+4.5236	− .3164	−2.4620	− .9419	− .1797	− .0172	
5.5	−1.5915	+6.2828	+3.4468	−4.0832	−3.0436	− .9013	− .1378	
6.0	−2.5302	+6.4536	+11.7834	−3.3613	−7.7458	−3.3690	− .8705	− .1361
6.5	−3.1324	+3.5506	+25.1749	+6.6506	−14.5558	−12.1739	−4.4488	−1.0093
7.0	−3.1059	−3.4974	+40.5401	+38.8142	−14.2714	−32.1868	−18.6006	−6.1044
							$^1D_k^{(1)}(z)$	
3.5	+ .7660	−1.2584	− .0823	+ .0565	+ .0074			
4.0	+1.2277	−1.8344	− .6849	+ .0815	+ .0304	+ .0051		
4.5	+1.6832	−1.8877	−2.1735	− .1481	+ .0840	+ .0260		
5.0	+1.9322	− .7766	−4.8160	−1.4141	+ .0598	+ .0853	− .0010	
5.5	+1.7750	+2.0529	−8.0904	−5.4407	− .7879	+ .1463	+ .0442	
6.0	+1.0840	+6.6706	−9.8249	−14.8121	−5.2657	− .4507	+ .1660	+ .1452
6.5	− .1295	+12.2448	−5.5858	−31.2672	−20.9246	−5.4457	− .1685	+ .2918
7.0	−1.6797	+16.8159	+10.8370	−50.8593	−62.7531	−29.3100	−5.7721	+ .0160

THE PRINCIPLE OF MINIMIZED ITERATIONS IN THE SOLUTION OF THE MATRIX EIGENVALUE PROBLEM[1]

BY

W. E. ARNOLDI

Hamilton Standard Division, United Aircraft Corporation, East Hartford, Conn.

An interpretation of Dr. Cornelius Lanczos' iteration method, which he has named "minimized iterations", is discussed in this article, expounding the method as applied to the solution of the characteristic matrix equations both in homogeneous and non-homogeneous form. This interpretation leads to a variation of the Lanczos procedure which may frequently be advantageous by virtue of reducing the volume of numerical work in practical applications. Both methods employ essentially the same algorithm, requiring the generation of a series of orthogonal functions through which a simple matrix equation of reduced order is established. The reduced matrix equation may be solved directly in terms of certain polynomial functions obtained in conjunction with the generated orthogonal functions, and the convergence of the solution may be observed as the order of the reduced matrix is successively increased with the order of the original matrix as a limit. The method of minimized iterations is recommended as a rapid means for determining a small number of the larger eigenvalues and modal columns of a large matrix and as a desirable alternative for various series expansions of the Fredholm problem.

1. The conventional iterative procedures. It is frequently required that real latent roots, or eigenvalues, and modal columns be determined for a real numerical matrix, u, of order, n, in the characteristic homogeneous equation,*

$$(\lambda I - u)k = 0 \tag{1}$$

which is satisfied by any of n values of the scalar, $\lambda = \lambda_r$ with their associated modal columns, $k = k_r$. Beginning with an arbitrary column, k_0 , and repeatedly premultiplying by u, the iterative procedure, $k_{i+1} = uk_i$, will converge to the modal column corresponding to the largest, or dominant, latent root. After obtaining this solution, the dominant mode may be removed by any of several methods, so that the next largest root of the original matrix becomes the dominant root of an altered matrix, whereupon the same procedure may be repeated to obtain the next root, and so on, until all desired roots and modal columns have been obtained. Since the accuracy of each root and modal column is dependent upon the accuracy with which each previous column has been determined, this method obviously requires a wasteful amount of labor if the initial roots and modes are not needed, as is often the case in practical applications, and convergence can be extremely slow if the roots are not widely dispersed.

The non-homogeneous equation, or Fredholm problem in matrix form, may be represented by

$$(\lambda I - u)k = q, \tag{2}$$

[1]Received May 15, 1950.

*The general notation used in this discussion to represent matrices, columns, rows, and matrix equations follows as closely as possible the conventions established in "Elementary Matrices", Frazer, Duncan and Collar, Cambridge, 1938. Brackets and braces are employed only when necessary to avoid confusion, e.g., to distinguish between the column, k_i , and the matrix of m such columns, $[k_i]$.

where λ and q are specified and a solution is desired for k. This solution may be obtained by direct treatment of the array as simultaneous equations, a common algorithm being the method of pivotal condensation [1], a lengthy and laborious procedure but sometimes indispensable. Among essentially iterative methods are the Schmidt expansion [2], which requires first the solution for latent roots, modal columns, and modal rows, whereupon k is found as

$$k = \sum_{r=1}^{n} \frac{1}{\lambda - \lambda_r} \frac{\kappa_r q}{\kappa_r k_r} k_r \qquad (\kappa_r = \text{modal row}).$$

Another iterative method is based upon the Liouville-Neumann expansion [2], where the reciprocal matrix is written as an infinite geometric series,

$$k = (\lambda I - u)^{-1} q = \lambda^{-1}(I + \lambda^{-1} u + \lambda^{-2} u^2 + \cdots) q$$

whose solution is obtained by iteration:

$$k_{i+1} = \lambda^{-1} q + \lambda^{-1} u k_i$$

Unfortunately, this method converges only if λ is greater in magnitude than the magnitude of the dominant root of u. It can also be applied in special cases when the field of roots can be shifted so as to meet the requirement for convergence, but the method is nevertheless of limited application and is often subject to slow convergence.

2. A classical method for reducing matrix order. The solution of the original homogeneous equation may be replaced by the solution of a matrix equation of reduced order by introducing the approximation that the modal column, k, may be represented by the sum of a series of arbitrary columns, k_i, multiplied by coefficients, c_i, to be established in the solution. Substituting

$$k = \sum_{i=1}^{m} c_i k_i \equiv [k_i]c \tag{3}$$

into Eq. (1) yields

$$(\lambda I - u)[k_i]c = 0, \tag{4}$$

where $[k_i]$ is a rectangular matrix whose columns are k_i, and c is a column of elements c_i. Now premultiply Eq. (4) by another rectangular matrix whose rows are κ_i, corresponding with k_i in such fashion that the row $\kappa = \sum_{i=1}^{m} c_i \kappa_i$ will satisfy the original matrix in the form,

$$\kappa(\lambda I - u) = 0.$$

It should be noted that the functions, k_i and κ_i, are also required to satisfy the point boundary conditions implicit in the matrix, u. The matrix equation now becomes

$$[\kappa_i](\lambda I - u)[k_i]c = 0,$$

or, in more concise form,

$$(\lambda I - [\kappa_i k_i]^{-1}[\kappa_i u k_i])c = 0. \tag{5}$$

Equation (5) will henceforth be referred to as the "reduced equation," and the matrix, $[\kappa_i k_i]^{-1}[\kappa_i u k_i]$, will be known as the "reduced matrix." Since m functions were assumed to be sufficient for an approximate solution, this equation involves a matrix of m order,

presumably less than the original n order, hence more rapidly solved than the original by conventional iterative procedures. If one could hope that m could be chosen sufficiently small, a solution might be available by direct expansion of the determinant, but it is usually necessary, in the general form given above, to employ a relatively large number of functions, k_i and κ_j, so that the advantages of this transformation of the original matrix are not universally apparent. Furthermore, the labor of calculating the elements of the reduced matrix may be great, particularly since the inversion of $[\kappa_j k_i]$ is required.

The Galerkin method [3] is a variation of the process outlined above, differing only in the choice of κ_j. By the Galerkin method, each κ_i is the transposed of the corresponding column, k_i, which somewhat simplifies the determination of the inverse matrix required. However, it is also objectionable on the basis of tedium.

3. The Lanczos method of minimized iterations. The following exposition differs in form from the discussion originally given by Lanczos [2], but it arrives at the same results and makes a further extension of the Galerkin type more evident. Lanczos reduces the matrix order as described above but eliminates the objections of labor in the formation of the reduced matrix, indicates a solution in terms of polynomial equations derived from a direct iterative procedure involving the original matrix, avoids the necessity for separately forming and solving the reduced matrix, and provides a means for efficiently generating the required columns and rows in order that a convergent solution will be obtained with a minimum number.

To deduce the Lanczos method of minimized iterations, it is first observed that the formation of the reduced matrix would be greatly simplified if the matrix, $[\kappa_j k_i]$, were of diagonal form, the otherwise arbitrary rows and columns initially chosen being of such form as to satisfy $\kappa_j k_i = 0$ when $i \neq j$. The inverse would then consist of a diagonal matrix whose individual elements would be the reciprocals of the corresponding elements in $[\kappa_j k_i]$, and the reduced equation would take the form

$$\left(\lambda I - \left[\frac{\kappa_j u k_i}{\kappa_j k_j}\right]\right) c = 0. \tag{6}$$

This form has further potential advantages, obtained by observing that the desired orthogonality relationship among the chosen rows and columns may be established by generating these rows and columns from an initial row and column according to the following expressions:

$$k_{i+1} = uk_i - \alpha_i k_i - \beta_{i-1} k_{i-1} - \gamma_{i-2} k_{i-2} - \delta_{i-3} k_{i-3} - \cdots, \tag{7}$$

$$\kappa_{i+1} = \kappa_i u - A_i \kappa_i - B_{i-1} \kappa_{i-1} - C_{i-2} \kappa_{i-2} - D_{i-3} \kappa_{i-3} - \cdots. \tag{8}$$

The necessary scalar constants, α_i, β_{i-1}, γ_{i-2}, $\cdots$, A_i, B_{i-1}, C_{i-2}, $\cdots$, may be defined by forming certain scalars as row-column products and noting that many elements vanish, due to the orthogonality relation postulated, $\kappa_j k_i = 0$ for $i \neq j$, as follows:

$$\kappa_i k_{i+1} = \kappa_i u k_i \quad - \alpha_i \kappa_i k_i = 0,$$

$$\kappa_i k_{i+2} = \kappa_i u k_{i+1} - \beta_i \kappa_i k_i = 0,$$

$$\kappa_i k_{i+3} = \kappa_i u k_{i+2} - \gamma_i \kappa_i k_i = 0,$$

etc.

also

$$\kappa_{i+1}k_i = \kappa_i u k_i \quad - A_i \kappa_i k_i = 0,$$

$$\kappa_{i+2}k_i = \kappa_{i+1} u k_i - B_i \kappa_i k_i = 0,$$

$$\kappa_{i+3}k_i = \kappa_{i+2} u k_i - C_i \kappa_i k_i = 0,$$

etc.

whence

$$\alpha_i = \frac{\kappa_i u k_i}{\kappa_i k_i}, \qquad \beta_i = \frac{\kappa_i u k_{i+1}}{\kappa_i k_i}, \qquad \gamma_i = \frac{\kappa_i u k_{i+2}}{\kappa_i k_i}, \qquad \text{etc.},$$

$$A_i = \frac{\kappa_i u k_i}{\kappa_i k_i}, \qquad B_i = \frac{\kappa_{i+1} u k_i}{\kappa_i k_i}, \qquad C_i = \frac{\kappa_{i+2} u k_i}{\kappa_i k_i}, \qquad \text{etc.}$$

Note that $\alpha_i = A_i$. Other identities may be found as follows:

$$uk_i = k_{i+1} + \alpha_i k_i + \beta_{i-1} k_{i-1} + \cdots ,$$

$$\kappa_{i+1} u k_i = \kappa_{i+1} k_{i+1} .$$

Also,

$$\kappa_i u = \kappa_{i+1} + A_i \kappa_i + B_{i-1} \kappa_{i-1} + \cdots ,$$

$$\kappa_i u k_{i+1} = \kappa_{i+1} k_{i+1} .$$

Therefore,

$$\beta_i = B_i .$$

The later constants will vanish, as is next shown.

$$\kappa_i u = \kappa_{i+1} + A_i \kappa_i + B_{i-1} \kappa_{i-1} + C_{i-2} \kappa_{i-2} + \cdots .$$

Therefore,

$$\kappa_i u k_{i+2} = 0$$

whence,

$$\gamma_i = 0.$$

Similarly, $C_i = 0$, and all further scalars, δ_i, ϵ_i, $\cdots$, D_i, E_i, $\cdots$, are also found to vanish.

The functional relationships by means of which the desired orthogonal rows and columns are generated thus become

$$k_{i+1} = uk_i - \alpha_i k_i - \beta_{i-1} k_{i-1} , \tag{9}$$

$$\kappa_{i+1} = \kappa_i u - \alpha_i \kappa_i - \beta_{i-1} \kappa_{i-1} , \tag{10}$$

where

$$\alpha_i = \frac{\kappa_i u k_i}{\kappa_i k_i}, \qquad \beta_{i-1} = \frac{\kappa_{i-1} u k_i}{\kappa_{i-1} k_{i-1}}.$$

The pertinent scalars and zeros having been established, it is now convenient to identify the elements of the reduced matrix, outlined as follows:

$$\left[\frac{\kappa_i u k_i}{\kappa_i k_i}\right] \equiv \left\|\begin{matrix} \frac{\kappa_1 u k_1}{\kappa_1 k_1} & \frac{\kappa_1 u k_2}{\kappa_1 k_1} & \frac{\kappa_1 u k_3}{\kappa_1 k_1} & \frac{\kappa_1 u k_4}{\kappa_1 k_1} & \cdots & \frac{\kappa_1 u k_m}{\kappa_1 k_1} \\ \frac{\kappa_2 u k_1}{\kappa_2 k_2} & \frac{\kappa_2 u k_2}{\kappa_2 k_2} & \frac{\kappa_2 u k_3}{\kappa_2 k_2} & \frac{\kappa_2 u k_4}{\kappa_2 k_2} & \cdots & \frac{\kappa_2 u k_m}{\kappa_2 k_2} \\ \frac{\kappa_3 u k_1}{\kappa_3 k_3} & \frac{\kappa_3 u k_2}{\kappa_3 k_3} & \frac{\kappa_3 u k_3}{\kappa_3 k_3} & \frac{\kappa_3 u k_4}{\kappa_3 k_3} & \cdots & \frac{\kappa_3 u k_m}{\kappa_3 k_3} \\ \frac{\kappa_4 u k_1}{\kappa_4 k_4} & \frac{\kappa_4 u k_2}{\kappa_4 k_4} & \frac{\kappa_4 u k_3}{\kappa_4 k_4} & \frac{\kappa_4 u k_4}{\kappa_4 k_4} & \cdots & \frac{\kappa_4 u k_m}{\kappa_4 k_4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\kappa_m u k_1}{\kappa_m k_m} & \frac{\kappa_m u k_2}{\kappa_m k_m} & \frac{\kappa_m u k_3}{\kappa_m k_m} & \frac{\kappa_m u k_4}{\kappa_m k_m} & \cdots & \frac{\kappa_m u k_m}{\kappa_m k_m} \end{matrix}\right\|. \tag{11}$$

The elements are easily recognized and replaced, yielding

$$\left[\frac{\kappa_i u k_i}{\kappa_i k_i}\right] \equiv \left\|\begin{matrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & \beta_2 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_3 & \beta_3 & \cdots & 0 \\ 0 & 0 & 1 & \alpha_4 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_m \end{matrix}\right\|. \tag{12}$$

The vanishing of so many elements makes the expansion of the characteristic determinant extremely simple, so that a treatment of the reduced matrix equation by conventional iterative methods is unnecessary, and the step from the formation of the orthogonal row and column sequence to the solution for roots and modal columns may be taken without specifically examining the reduced matrix. The determinant will be expanded in stages, beginning with first order, which corresponds to the choice of one row and column, and successively increasing the order until the desired number of rows and columns has been reached. The resulting polynomials, individually equated to zero, will

have roots which will represent successive approximations to the roots of the original matrix. These polynomials, which may be written by inspection, follow:
The determinant is

$$\left| \lambda I - \left[\frac{\kappa_j u k_i}{\kappa_j k_j} \right] \right| = \begin{vmatrix} \lambda - \alpha_1 & -\beta_1 & 0 & \cdots \\ -1 & \lambda - \alpha_2 & -\beta_2 & \cdots \\ 0 & -1 & \lambda - \alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = p_m(\lambda). \tag{13}$$

The polynomials are

$$\begin{aligned} p_1(\lambda) &= \lambda - \alpha_1 , \\ p_2(\lambda) &= (\lambda - \alpha_2)p_1(\lambda) - \beta_1 , \\ p_3(\lambda) &= (\lambda - \alpha_3)p_2(\lambda) - \beta_2 p_1(\lambda), \\ p_4(\lambda) &= (\lambda - \alpha_4)p_3(\lambda) - \beta_3 p_2(\lambda), \\ p_{i+1}(\lambda) &= (\lambda - \alpha_{i+1})p_i(\lambda) - \beta_i p_{i-1}(\lambda). \end{aligned} \tag{14}$$

The solution for the modal column corresponding to a root of $p_m(\lambda) = 0$ will be available after the coefficients, c_i , have been obtained by direct solution of the reduced matrix equation as a set of simultaneous scalar equations. Since one coefficient may be arbitrary, let $c_1 = 1$. Then,

$$\begin{aligned} c_2 &= \frac{\lambda - \alpha_1}{\beta_1} &&= \frac{p_1(\lambda)}{\beta_1} , \\ c_3 &= \frac{(\lambda - \alpha_2)c_2 - 1}{\beta_2} &&= \frac{p_2(\lambda)}{\beta_1\beta_2} , \\ c_4 &= \frac{(\lambda - \alpha_3)c_3 - c_2}{\beta_3} &&= \frac{p_3(\lambda)}{\beta_1\beta_2\beta_3} , \\ c_{i+1} &= \frac{(\lambda - \alpha_i)c_i - c_{i-1}}{\beta_i} &&= \frac{p_i(\lambda)}{\beta_1\beta_2 \cdots \beta_i} , \end{aligned}$$

and we have the modal column through its initial definition,

$$k = \sum_{i=1}^{m} c_i k_i = k_1 + \frac{p_1(\lambda)}{\beta_1} k_2 + \frac{p_2(\lambda)}{\beta_1\beta_2} k_3 + \cdots + \frac{p_{m-1}(\lambda)}{\beta_1\beta_2 \cdots \beta_{m-1}} k_m . \tag{15}$$

It is important to note that, not only is this a finite series, but it is limited in length to a maximum number of terms, $m = n$, the order of the original matrix. The generation of the orthogonal function sequence stops at this point, since the original matrix on

which it is based is incapable of defining more than n linearly independent functions, and when this number has been reached the reduced matrix will yield precisely the roots and modal columns of the original matrix. This characteristic, more fully discussed by Lanczos [2], is an important feature of the method of minimized iterations, since it indicates that this method, unlike the conventional matrix iteration algorithm, will never require a continuation ad infinitum.

4. Minimized iterations for the non-homogeneous equation. Given the equation

$$(\lambda I - u)k = q, \tag{16}$$

where λ and q are specified and a solution for k is desired, the reduced equation is formed exactly as for the homogeneous equation, replacing k and premultiplying by $[\kappa_j]$ as follows:

$$\left(\lambda I - \left[\frac{\kappa_j u k_i}{\kappa_j k_j}\right]\right)c = \left[\frac{\kappa_j}{\kappa_j k_j}\right]q.$$

The column on the right side of the equation is vastly simplified by letting $k_1 = q$ whence

$$\left(\lambda I - \left[\frac{\kappa_j u k_i}{\kappa_j k_j}\right]\right)c = \{1, 0, 0, \cdots 0\}. \tag{17}$$

The method of minimized iterations is now employed to determine the same scalars, α_i and β_i, as before, whereupon the solution for c is indicated as follows:

$$c = \left(\lambda I - \left[\frac{\kappa_j u k_i}{\kappa_j k_j}\right]\right)^{-1}\{1, 0, 0, \cdots 0\}. \tag{18}$$

The inverse of a matrix is defined as the quotient of the adjoint of the matrix by its determinant, and the adjoint is the transpose of a matrix whose elements are the cofactors of the corresponding elements in the matrix to be inverted. The determinant of the matrix is already available as $p_m(\lambda)$, so it is required to examine the adjoint. Since the adjoint is used to premultiply a column whose elements all vanish except the first, only the first row of the adjoint is needed. The elements of this row will be the cofactors of the elements of the first column in the determinant array of the reduced equation, (13). These cofactors can be written directly by inspection, using polynomials of the form already defined in Eq. (14) but in reverse order, beginning with the lower right corner of the array instead of the upper left. These reversed polynomials will be designated as

$$\begin{aligned}
\bar{p}_1(\lambda) &= \lambda - \alpha_m\,, \\
\bar{p}_2(\lambda) &= (\lambda - \alpha_{m-1})\bar{p}_1(\lambda) - \beta_{m-1}\,, \\
\bar{p}_3(\lambda) &= (\lambda - \alpha_{m-2})\bar{p}_2(\lambda) - \beta_{m-2}\bar{p}_1(\lambda), \\
\bar{p}_{i+1}(\lambda) &= (\lambda - \alpha_{m-i})\bar{p}_i(\lambda) - \beta_{m-i}\bar{p}_{i-1}(\lambda).
\end{aligned} \tag{19}$$

Using these, the first cofactor is $\bar{p}_{m-1}(\lambda)$, the second is $\beta_1\bar{p}_{m-2}(\lambda)$, the third is $\beta_1\beta_2\bar{p}_{m-3}(\lambda)$, and the coefficients, c_i, may be written,

$$c_1 = \frac{\bar{p}_{m-1}(\lambda)}{p_m(\lambda)},$$

$$c_2 = \beta_1 \frac{\bar{p}_{m-2}(\lambda)}{p_m(\lambda)},$$

$$c_3 = \beta_1\beta_2 \frac{\bar{p}_{m-3}(\lambda)}{p_m(\lambda)},$$

$$c_i = \beta_1\beta_2 \cdots \beta_{i-1} \frac{\bar{p}_{m-i}(\lambda)}{p_m(\lambda)}.$$

Applying these to the definition of k, the solution is complete.

$$k = \sum_{i=1}^{m} c_i k_i = \frac{1}{p_m(\lambda)} (\bar{p}_{m-1}(\lambda)k_1 + \beta_1\bar{p}_{m-2}(\lambda)k_2 + \cdots + \beta_1\beta_2 \cdots \beta_{m-1}k_m). \tag{20}$$

5. An outline of the Lanczos algorithm. While the preceding derivation may appear to be complicated, practical application of minimized iterations to numerical calculations requires very simple repeated steps, which are outlined below and which may easily be translated into operations on automatic computing machinery.

First stage: Given κ_1 and k_1.

Form $\kappa_1 u$, uk_1, $\kappa_1 k_1$.

Form $\kappa_1 uk_1$.

Compute $\alpha_1 = \dfrac{\kappa_1 uk_1}{\kappa_1 k_1}$.

$$p_1(\lambda) = \lambda - \alpha_1 = 0$$

now yields a first approximation to the first root.

Second stage: Form $\kappa_2 = \kappa_1 u - \alpha_1 \kappa_1$, $k_2 = uk_1 - \alpha_1 k_1$.

Form $\kappa_2 u$, uk_2, $\kappa_2 k_2$, $\kappa_2 uk_2$, $\kappa_1 uk_2$.

Compute $\alpha_2 = \dfrac{\kappa_2 uk_2}{\kappa_2 k_2}$, $\beta_1 = \dfrac{\kappa_1 uk_2}{\kappa_1 k_1}$.

$$p_2(\lambda) = (\lambda - \alpha_2)p_1(\lambda) - \beta_1 = 0$$

gives a second approximation to the first root and a first approximation to the second.

Third stage: Form $\kappa_3 = \kappa_2 u - \alpha_2 \kappa_2 - \beta_1 \kappa_1$, $k_3 = uk_2 - \alpha_2 k_2 - \beta_1 k_1$.

Form $\kappa_3 u$, uk_3, $\kappa_3 k_3$, $\kappa_3 uk_3$, $\kappa_2 uk_3$.

$$\text{Compute } \alpha_3 = \frac{\kappa_3 u k_3}{\kappa_3 k_3}, \ \beta_2 = \frac{\kappa_2 u k_3}{\kappa_2 k_2}.$$

$$p_3(\lambda) = (\lambda - \alpha_3)p_2(\lambda) - \beta_2 p_1(\lambda) = 0$$

gives the third approximation to the first root, the second approximation to the second root, and a first approximation to the third root. The calculation is continued as far as necessary in continued stages. While the process of minimized iterations as described here can be expected eventually to provide the desired latent roots to any desired degree of accuracy with any initially chosen row and column, the rate of convergence can be made extremely rapid by first multiplying the initial row and column several times by the matrix, u, and then beginning the application of the method. The advantage of doing this is that the minimized iterations will then begin with functions whose normal mode components will be "ordered" in the sense that the dominant mode will represent a large part of the initial functions, the second mode will be somewhat smaller but next in importance, the third mode will follow, and so on. To a rough degree of approximation, the orthogonal functions thus generated will then in turn approximate the normal modes in the same order, and the roots of the successive polynomials will also appear in this order. Lanczos [2] presents an impressive numerical example which demonstrates extremely powerful convergence.

It is convenient, particularly in the use of punched card machines, to form certain of the required products simultaneously and to supplement the required operations by check calculations which verify the orthogonality of the generated rows and columns. As an example, consider the third stage. The product, $\kappa_3 u$, is indicated first and the matrix is bordered by several columns as follows:

$$\kappa_3[u, k_1, k_2, k_3] = [\kappa_3 u, 0, 0, \kappa_3 k_3].$$

The two zeroes serve as a numerical check. Likewise, the product, uk_3, is also formed in conjunction with others by bordering u with several rows, as follows:

$$[u, \kappa_1 u, \kappa_2 u, \kappa_3 u]k_3 = \{uk_3, 0, \kappa_2 u k_3, \kappa_3 u k_3\}.$$

The zero here serves also as a check, and all the scalar products required for the computation of α_3 and β_2 are available from these operations. In numerical calculations, when the zeroes may occur as small finite numbers due to rounding errors, a refinement is possible by using these numbers to determine the constants, γ_i, δ_i, ϵ_i, etc. of Eq. (7) and (8), which may be employed as indicated by these equations for the generation of further orthogonal functions.

6. Minimized iterations in the Galerkin method. It will now be shown that a minimized iteration technique, similar in form but slightly different in detail, is also applicable to the solution of the eigenvalue problem by the Galerkin method, and that this method may offer certain computational advantages. Beginning with the reduced equation (5) in Galerkin form, but with $k_i' k_i = 0$ for $i \neq j$,

$$\left(\lambda I - \left[\frac{k_i' u k_i}{k_i' k_i}\right]\right)c = 0 \tag{21}$$

the generation of suitable columns, k_i, will be detailed. Using the same technique proposed by Lanczos, let

$$k_{i+1} = uk_i - \alpha_i k_i - \beta_{i-1} k_{i-1} - \gamma_{i-2} k_{i-2} - \delta_{i-3} k_{i-3} \cdots \tag{22}$$

and solve for the constants required on the basis that $k_j' k_i = 0$ $(i \neq j)$.

$$k_i' k_{i+1} = k_i' u k_i \quad - \alpha_i k_i' k_i = 0,$$

$$k_i' k_{i+2} = k_i' u k_{i+1} - \beta_i k_i' k_i = 0,$$

$$k_i' k_{i+3} = k_i' u k_{i+2} - \gamma_i k_i' k_i = 0,$$

whence

$$\alpha_i = \frac{k_i' u k_i}{k_i' k_i}, \qquad \beta_i = \frac{k_i' u k_{i+1}}{k_i' k_i}, \qquad \gamma_i = \frac{k_i' u k_{i+2}}{k_i' k_i}, \qquad \text{etc.}$$

However, in the Galerkin case none of these constants vanish. Nevertheless,

$$k_{i+1}' u k_i = k_{i+1}' k_{i+1} ,$$

and

$$k_{i+j}' u k_i = 0 \qquad \text{if} \qquad j > 1.$$

Thus, the reduced matrix becomes

$$\left[\frac{k_j' u k_i}{k_j' k_j}\right] \equiv \left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \cdots & \\ 1 & \alpha_2 & \beta_2 & \gamma_2 & \cdots & \\ 0 & 1 & \alpha_3 & \beta_3 & \cdots & \\ 0 & 0 & 1 & \alpha_4 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_m \end{matrix} \right\|, \tag{23}$$

which is of the same form as Eq. (11), (compare with Eq. (12)), and the polynomial expansions of the determinant by steps are, beginning with the definition,

$$\left| \lambda I - \left[\frac{k_j' u k_i}{k_j' k_j}\right] \right| = \left| \begin{matrix} \lambda - \alpha_1 & -\beta_1 & -\gamma_1 & \cdots \\ -1 & \lambda - \alpha_2 & -\beta_2 & \cdots \\ 0 & -1 & \lambda - \alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix} \right| = p_m(\lambda), \tag{24}$$

$$\begin{aligned} p_1(\lambda) &= \lambda - \alpha_1 , \\ p_2(\lambda) &= (\lambda - \alpha_2) p_1(\lambda) - \beta_1 , \\ p_3(\lambda) &= (\lambda - \alpha_3) p_2(\lambda) - \beta_2 p_1(\lambda) - \gamma_1 , \\ p_4(\lambda) &= (\lambda - \alpha_4) p_3(\lambda) - \beta_3 p_2(\lambda) - \gamma_2 p_1(\lambda) - \delta_1 , \end{aligned} \tag{25}$$

etc.

The roots of these polynomials, as in the Lanczos method, yield successive approximations to the latent roots of the original matrix.

The solution for the coefficients, c_i, required in order to define the modal columns, must here be accomplished in reverse order, due to the triangular form of the reduced matrix. These coefficients will be expressed in terms of a set of reversed polynomials, defined as follows:

Let $$c_m = 1,$$

$$c_{m-1} = \lambda - \alpha_m = \bar{p}_1(\lambda),$$

$$c_{m-2} = (\lambda - \alpha_{m-1})c_{m-1} - \beta_{m-1} = (\lambda - \alpha_{m-1})\bar{p}_1(\lambda) - \beta_{m-1} = \bar{p}_2(\lambda),$$

$$c_{m-3} = (\lambda - \alpha_{m-2})c_{m-2} - \beta_{m-2}c_{m-1} - \gamma_{m-2},$$

$$= (\lambda - \alpha_{m-2})\bar{p}_2(\lambda) - \beta_{m-2}\bar{p}_1(\lambda) - \gamma_{m-2} = \bar{p}_3(\lambda),$$

etc.

Hence, the modal column is written for λ corresponding to a root of $p_m(\lambda) = 0$, in terms of the reversed polynomials defined above.

$$k = \sum_{i=1}^{m} c_i k_i = \bar{p}_{m-1}(\lambda)k_1 + \bar{p}_{m-2}(\lambda)k_2 + \cdots + \bar{p}_1(\lambda)k_{m-1} + k_m. \tag{26}$$

It is interesting to compare Eqs. (21)-(26) with their corresponding relationships in the Lanczos method, Eqs. (6)-(15).

7. The Galerkin treatment of the non-homogeneous equation. In a manner similar to that applied by Lanczos and described in Sec. 4 for the solution of the non-homogeneous equation by minimized iterations, the method of minimized iterations may be extended to the solution of the reduced equation obtained by the Galerkin method. The necessary scalars are determined as described in Sec. 6, thus permitting the determination of the two sequences of polynomials, $p_i(\lambda)$ and $\bar{p}_i(\lambda)$. The next step is to solve the reduced equation for the coefficients, c_i. To accomplish this by determining the cofactors required from the adjoint of the matrix to be inverted is not convenient, in view of the triangular matrix form, but the solution can be easily obtained by first solving for the coefficients in terms of the last, c_m, working upward in the array until the use of $m - 1$ equations provide c_1 through c_{m-1} in terms of c_m, and then using the first equation to evaluate c_m. This process follows:

$$c_{m-1} = (\lambda - \alpha_m)c_m = \bar{p}_1(\lambda)c_m,$$

$$c_{m-2} = (\lambda - \alpha_{m-1})c_{m-1} - \beta_{m-1}c_m = \bar{p}_2(\lambda)c_m,$$

$$c_{m-3} = (\lambda - \alpha_{m-2})c_{m-2} - \beta_{m-2}c_{m-1} - \gamma_{m-2}c_m = \bar{p}_3(\lambda)c_m,$$

etc.

$$[(\lambda - \alpha_1)\bar{p}_{m-1}(\lambda) - \beta_1\bar{p}_{m-2}(\lambda) - \gamma_1\bar{p}_{m-3}(\lambda) \cdots]c_m = 1.$$

The coefficients having thus been determined, the solution follows directly.

$$k = \sum_{i=1}^{m} c_i k_i = \frac{\bar{p}_{m-1}(\lambda)k_1 + \bar{p}_{m-2}(\lambda)k_2 + \cdots + \bar{p}_1(\lambda)k_{m-1} + k_m}{(\lambda - \alpha_1)\bar{p}_{m-1}(\lambda) - \beta_1\bar{p}_{m-2}(\lambda) - \gamma_1\bar{p}_{m-3}(\lambda) \cdots} . \tag{27}$$

In view of the means by which the c_i were determined, Eq. (27) does not closely resemble Eq. (20).

8. An outline of the minimized iteration algorithm in the Galerkin method. Again it is desirable to present an outline of the actual steps in numerical calculation by minimized iterations, from which a logical sequence of operations may be established for purposes of automatic calculation.

First stage: Given k_1 .

Form uk_1 , $k_1'k_1$.

Form $k_1'uk_1$.

Compute $\alpha_1 = \dfrac{k_1'uk_1}{k_1'k_1}$.

$p_1(\lambda) = \lambda - \alpha_1$.

Second stage: Form $k_2 = uk_1 - \alpha_1 k_1$.

Form uk_2 , $k_2'k_2$, $k_2'uk_2$, $k_1'uk_2$.

Compute $\alpha_2 = \dfrac{k_2'uk_2}{k_2'k_2}$, $\beta_1 = \dfrac{k_1'uk_2}{k_1'k_1}$.

$p_2(\lambda) = (\lambda - \alpha_2)p_1(\lambda) - \beta_1$.

Third stage: Form $k_3 = uk_2 - \alpha_2 k_2 - \beta_1 k_1$.

Form uk_3 , $k_3'k_3$, $k_3'uk_3$, $k_2'uk_3$, $k_1'uk_3$.

Compute $\alpha_3 = \dfrac{k_3'uk_3}{k_3'k_3}$, $\beta_2 = \dfrac{k_2'uk_3}{k_2'k_2}$, $\gamma_1 = \dfrac{k_1'uk_3}{k_1'k_1}$.

$p_3(\lambda) = (\lambda - \alpha_3)p_2(\lambda) - \beta_2 p_1(\lambda) - \gamma_1$.

As in the Lanczos procedure described in Sec. 5, it is advantageous to precede this process with several premultiplications of an arbitrary column in order to control the relative magnitudes of the various modal components of the column, k_1 , so that the roots will be obtained from the successive polynomials in order of relative magnitudes. As indicated for the Lanczos algorithm, it is convenient in punched card procedures to form certain of the required products simultaneously and to include check calculations.

To this end the matrix, u, is bordered by a sequence of rows, as exemplified below for the third stage.

$$[u, k_1', k_2', k_3']k_3 = \{uk_3, 0, 0, k_3'k_3\}.$$

The two zeroes serve as a check on the orthogonality of the generated columns. In addition, the following operations are necessary:

$$[k_3', k_2', k_1']uk_3 = \{k_3'uk_3, k_2'uk_3, k_1'uk_3\}.$$

whence the constants, α_3, β_2, and γ_1, may be calculated.

9. General comments on minimized iterations. It should be noted that the foregoing presentation and extension of the method originally proposed by Lanczos has leaned toward the pictorial approach to a means of mathematical analysis rather than concerning itself with important considerations of mathematical rigor. The special cases of equal roots and of unfortunate choice of arbitrary row or column, where a deficiency in modal components might lead to peculiarities in the results, have been deliberately ignored, first, because these cases are adequately treated in Lanczos' paper [2], and second, because they would detract from the object of presenting a practical procedure and of emphasizing the computational aspects which are of interest to the applied mathematician and to the engineer. Furthermore, in the interest of providing a simple derivation which would admit of obvious extension to the Galerkin type equation, the original intent of the name, *minimized* iterations, was passed over lightly, deriving the basic scalars through strictly algebraic operations without consideration of their more fundamental purpose of providing the maximum utility with the minimum number of row and column functions. The advantages of orthogonal functions in computation provided an equally direct reason for the choice of these scalars.

The method of minimized iterations, in either the Lanczos or the Galerkin variations, is recommended as an alternative to conventional procedures in the numerical solution of matrix problems on the basis of potentially large time-saving in computation. Its advantages are most evident when the determination of a number of eigenvalues, or latent roots, is of prime interest, and the determination of modal columns or the solution of non-homogeneous equations can also be greatly expedited, subject to an efficient organization of problems for whatever type of computing machinery may be available.

The Galerkin variation appears to be of greatest advantage in the eigenvalue problem, requiring substantially only half the number of matrix-column products involved in the Lanczos procedure. However, the computation of modal columns by this method is somewhat more cumbersome, particularly if successive approximations to a modal column are to be studied, in view of the requirement for reversed polynomials. In the non-homogeneous equation, on the other hand, this disadvantage applies to both versions, the only significant difference being that the Galerkin polynomials are longer.

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SOURCE-SUPERPOSITION METHOD OF SOLUTION OF A PERIODICALLY OSCILLATING WING AT SUPERSONIC SPEEDS*

BY

H. J. STEWART AND TING-YI LI

California Institute of Technology

Introduction and summary. In a recent paper Evvard (Ref. 1) discussed the linearized theory of the non-steady motion of three dimensional wings by methods which he had previously developed for the treatment of the corresponding steady flow problems (Refs. 2 and 3). Evvard represented the wing by a distribution of sources, and the important result of his steady state theory concerned the determination of the flow in a region influenced by a subsonic leading edge or wing tip. He showed that the influence of the flow around this subsonic edge of a flat lifting wing on the velocity potential at

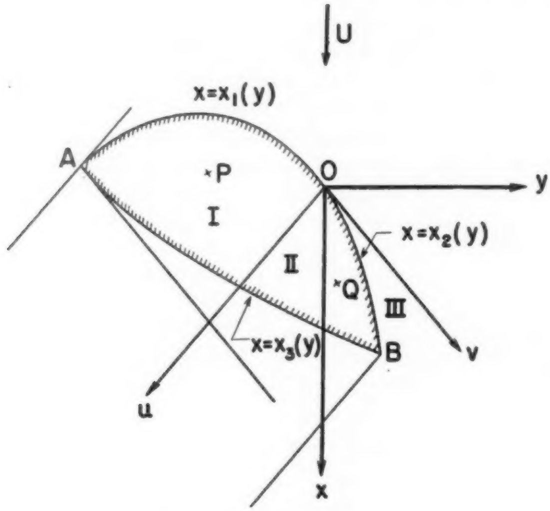

FIG. 1. Geometry of Wing.

a point within the region of influence of this edge is exactly equal to and of opposite sign to the contribution to the potential from the sources distributed over a simply determined region of the wing. In his paper on non-steady motion, he was able by similar methods to determine an explicit formula for the velocity potential; however he could not express the results in a similar, "equivalent area", form.

The present paper is concerned with the same problem of the non-steady lift of finite wings at supersonic speeds, particularly in regions which are influenced by subsonic leading edges or wing tips. It is shown that the simple "equivalent area" theorem developed by Evvard for the steady state case is also valid for oscillating wings. The

*Received December 30, 1949.

theorem is not extended to arbitrary non-steady motions, and an example where the theorem in this simple form is apparently not valid is demonstrated.

The basic differential equation and boundary conditions. Consider a wing in a steady supersonic flow of velocity U and Mach number M in the direction of the x-axis. Then the velocity potential φ which governs any small, possibly non-steady, disturbance produced by the wing satisfies the linearized differential equation

$$\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{2U}{a^2}\frac{\partial^2\varphi}{\partial x\,\partial t} + \left(\frac{U^2}{a^2} - 1\right)\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}, \tag{1}$$

where (x, y, z) are cartesian coordinates, t is the time variable and a is the speed of sound in the undisturbed flow so that $U = Ma$. The wing is assumed to be near the plane $z = 0$.

With the approximation of the linear equation, it is permissible to replace the boundary conditions at the point (x, y, z) on the actual wing surface by the same boundary conditions applied on the plane $z = 0$ at the point $(x, y, 0)$. In order to express the boundary conditions it is necessary in general to divide the wing surface into two different types of regions (see Fig. 1). The origin of the coordinate system is taken at the point 0 where the Mach line $0v$ is tangent to the leading edge. The leading edge is thus divided into two segments, the segment $0A$ which is defined by $x = x_1(y)$ or $y = y_1(x)$ and is a supersonic leading edge and the segment $0B$ which is defined by $x = x_2(y)$ or $y = y_2(x)$ and is a subsonic leading edge. As a matter of convenience it is assumed that the trailing edge, $x = x_3(y)$, is a supersonic trailing edge where the Kutta condition need not apply. The Mach line $0u$ then divides the wing into two regions. Region I, which is bounded by $x = x_1(y)$, $x = x_3(y)$ and $0u$, may be referred to as a purely supersonic region. Region II, which is bounded by $x = x_2(y)$, $x = x_3(y)$ and $0u$ may be referred to as a mixed supersonic region (Ref. 4).

At any point on the surface of the wing the flow must be tangential to the surface at any instant. This boundary condition, linearized, and applied to an oscillating condition is

$$\frac{\partial\varphi_T}{\partial z} = w_T(x, y, +0)\exp(i\nu t) = U\Lambda_T(x, y, +0)\exp(i\nu t), \tag{2}$$

$$\frac{\partial\varphi_B}{\partial z} = w_B(x, y, -0)\exp(i\nu t) = -U\Lambda_B(x, y, -0)\exp(i\nu t), \tag{2a}$$

where, except for the time factor, $w(x, y, z)$ is the z component of the velocity and Λ is the effective slope of the streamline and ν is the frequency of oscillation. The subscript T refers to the top of the wing and the subscript B refers to the bottom of the wing. In general w_T and w_B (or Λ_T and Λ_B) need not be related. A sign convention for Λ_T and Λ_B, adopted in Ref. 1 is also used here and is shown in Fig. 2.

From the definition of a purely supersonic region, there can be no disturbance in the flow ahead of the line $x = x_1(y)$. For any point P in Region I the velocity w is thus known at every point on the plane $z = 0$ in the forward Mach cone from the point P. On the wing w is given by Eq. 2 or Eq. 2a, ahead of the wing $w = 0$.

For a point Q in Region II conditions are more complex. As before, the velocity w is given for that portion of the plane $z = 0$ in the forward Mach cone from Q which is

covered by the wing by Eq. 2 or 2a. Also $w = 0$ and $\varphi = 0$ ahead of the line segments $x = x_1(y)$ and $0v$. Since $x = x_2(y)$ is a subsonic leading edge, there is, in general, an interaction between the upper and lower surfaces which produces an upwash across the plane $z = 0$ in Region III which is bounded by $x = x_2(y)$ and $0v$. This upwash cannot, in general, be specified in advance. For this region the pressure must be continuous across the plane $z = 0$ so the linearized boundary condition for this region is thus

$$\frac{\partial \varphi_T}{\partial t} + U \frac{\partial \varphi_T}{\partial x} = \frac{\partial \varphi_B}{\partial t} + U \frac{\partial \varphi_B}{\partial x}. \tag{3}$$

The boundary conditions on the plane $z = 0$ for a point Q in Region II are thus of a mixed type, involving w over the wing, pressure over Region III and no disturbance ahead of the lines $0v$ and $x = x_1(y)$.

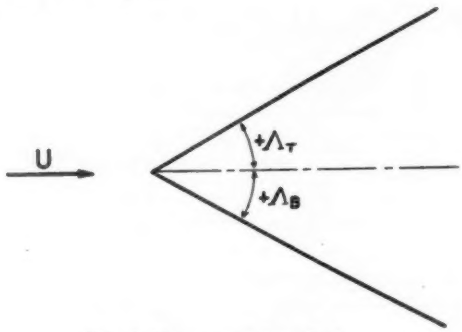

FIG. 2. Sign convention of Λ's.

Elementary oscillating source potential. Elementary solutions of Eq. 1 which can be superimposed to obtain more complex solutions can easily be obtained by the method of separation of variables. For this purpose it is convenient to introduce the following coordinate transformation:

$$\begin{aligned} r &= [x^2 - \beta^2(y^2 + z^2)]^{1/2}, \\ \mu &= \left[1 - \frac{\beta^2(y^2 + z^2)}{x^2}\right]^{-1/2}, \\ \omega &= \tan^{-1}(z/y), \qquad \tau = \beta^2 a\left(t - \frac{Mx}{\beta^2 a}\right), \end{aligned} \tag{4}$$

where $\beta^2 = M^2 - 1$.

These space variables were found useful in the treatment of steady conical flows (Ref. 5). The time transformation is similar to a combined Lorentz and Galilean transformation and has been used by Miles (Ref. 6). In these coordinates Eq. 1 is

$$r^2 \frac{\partial^2 \varphi}{\partial \tau^2} = r^2 \frac{\partial^2 \varphi}{\partial r^2} + 2r \frac{\partial \varphi}{\partial r} + \frac{\partial}{\partial \mu}\left[(1 - \mu^2)\frac{\partial \varphi}{\partial \mu}\right] + \frac{1}{1 - \mu^2}\frac{\partial^2 \varphi}{\partial \omega^2}. \tag{5}$$

This is identical with the form of the classical wave equation in spherical coordinates.

The solutions of Eq. 5, obtained by the method of separation of variables are

$$\varphi = \sum_{l,m,n} A_{lmn} \begin{Bmatrix} \cos m\omega \\ \sin m\omega \end{Bmatrix} \begin{Bmatrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{Bmatrix} \begin{Bmatrix} r^{-1/2} J_{-n-1/2}(lr) \\ r^{-1/2} J_{n+1/2}(lr) \end{Bmatrix} \exp(\pm il\tau), \tag{6}$$

where l, m and n are the separation parameters. $P_n^m(\mu)$ and $Q_n^m(\mu)$ are Associated Legendre functions and $J_{\pm(n+1/2)}(lr)$ is a Bessel function of order $\pm(n + 1/2)$.

For $m = n = 0$, a simple solution of Eq. 6 is

$$\varphi_1 = Ar^{-1/2} J_{-1/2}(lr) \exp(il\tau). \tag{7}$$

If τ is replaced by the physical time variable from Eq. 4, the Bessel function is written in the trigonometric form and l is eliminated by the relation $\nu = l\beta^2 a$, Eq. 7 becomes

$$\varphi_1 = \frac{A_1}{r} \cos\left(\frac{\nu r}{\beta^2 a}\right) \exp\left[i\nu\left(t - \frac{Ux}{\beta^2 a^2}\right)\right], \tag{8}$$

where A_1 is a new arbitrary constant. Equation 8 may be considered as defining a supersonic oscillating source. This basic solution has been used in this form by Miles (Ref. 7). If the basic solutions used by Garrick and Rubinow (Ref. 4) or by Evvard (Ref. 1) are applied to oscillating problems, they can be reduced to this same form. It may be noted that for $\nu = 0$ Eq. 8 reduces to the usual steady state source potential. The complete velocity potential field for an oscillating source is defined by Eq. 8 in the downstream Mach cone and as zero outside the Mach cone.

Velocity potential of an oscillating wing. For a point in the purely supersonic region the velocity potential due to the wing can readily be obtained by replacing the wing

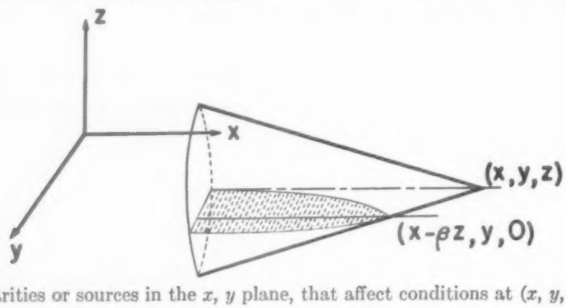

FIG. 3. Singularities or sources in the x, y plane, that affect conditions at (x, y, z) at instant t.

by a distribution of sources over the wing surface. If the region of dependence of a given point includes only that portion of the wing which is purely supersonic, the velocity potential for $z \geq 0$ due to the source distribution is thus

$$\Phi(x, y, z, t) = \iint_S A_T(\xi, \eta) \exp\left\{i\nu\left[t - \frac{U(x-\xi)}{\beta^2 a^2}\right]\right\} \cos\left(\frac{\nu r_1}{\beta^2 a}\right) \frac{d\xi\, d\eta}{r_1}, \tag{9}$$

where

$$r_1 = [(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2]^{1/2}. \tag{10}$$

Here $A_T(\xi, \eta)$ is the source strength per unit area at the coordinate (ξ, η) on the wing surface. The region of dependence, which determines the region of integration on the wing surface, is bounded on the downstream side by the line $r_1 = 0$ (Fig. 3).

If the source strength $A_T(\xi, \eta)$ can be chosen so the boundary conditions for the purely supersonic region are satisfied, Eq. 9 is the proper solution. In order to do this it is convenient to replace the integration variable η by

$$\beta(y - \eta) = -r_2 \sin \theta, \tag{11}$$

where

$$r_2 = [(x - \xi)^2 - \beta^2 z^2]^{1/2}. \tag{12}$$

With this notation Eq. 9 becomes

$$\begin{aligned}\Phi(x, y, z, t) = \frac{1}{\beta} \exp (i\nu t) \int_{\xi_1}^{x-\beta z} \exp \left[-\frac{i\nu U}{\beta^2 a^2} (x - \xi) \right] d\xi \\ \times \int_{-\pi/2}^{\pi/2} A_T\left(\xi, y + \frac{r_2}{\beta} \sin \theta\right) \cos \left(\frac{\nu r_2}{\beta^2 a} \cos \theta\right) d\theta,\end{aligned} \tag{13}$$

where ξ_1 is the least value of ξ on the leading edge. Since

$$\frac{\partial r_2}{\partial z} = -\frac{\beta^2 z}{r_2}, \tag{14}$$

$$\begin{aligned}\frac{\partial \Phi}{\partial z} = -\pi A_T(x - \beta z, y) \exp (i\nu t) \\ - \beta z \exp (i\nu t) \int_{\xi_1}^{x-\beta z} \exp \left[-\frac{i\nu U}{\beta^2 a^2} (x - \xi) \right] \frac{1}{r_2} d\xi \\ \times \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial r_2} \left[A_T\left(\xi, y + \frac{r_2}{\beta} \sin \theta\right) \cos \left(\frac{\nu r_2}{\beta^2 a} \cos \theta\right) \right] d\theta.\end{aligned} \tag{15}$$

If the function $A_T(\xi, \eta)$ is continuous in the neighborhood of the point (x, y), the magnitude of the double integral in Eq. 15 is finite for sufficiently small values of z; so

$$\lim_{z \to +0} \left(\frac{\partial \Phi}{\partial z}\right) = -\pi A_T(x, y) \exp (i\nu t). \tag{16}$$

By comparison of Eqs. 16 and 2, it is seen that

$$A_T(x, y) = -\frac{1}{\pi} w_T(x, y, +0) = -\frac{U}{\pi} \Lambda_T(x, y, +0). \tag{17}$$

For a point below the wing, $z \leq 0$, a similar analysis shows that

$$A_B(x, y) = \frac{1}{\pi} w_B(x, y, -0) = -\frac{U}{\pi} \Lambda_B(x, y, -0) \tag{17a}$$

The required source strength, $A(x, y)$ in the plane $z = 0$ is thus completely determined for any point in the purely supersonic region. On the wing $A(x, y)$ is given by Eq. 17 or Eq. 17a. Ahead of the wing the disturbance $(\partial \Phi / \partial z)_{z=0}$ is zero; so $A(x, y) = 0$ in this region. With these values of $A(x, y)$, Eq. 9 defines the velocity potential and thus the velocity components and the pressure on the wing. This analysis was given in a similar form by Miles in Ref. 6 (some errors in his presentation were corrected in Ref. 7).

The results of this analysis may be summarized in the following two theorems:

Theorem 1. The strength of the source at any point at any instant on the surface of an oscillating wing is linearly dependent on the downwash at that point and at that instant and is independent of the downwash of the neighboring points.

Theorem 2. The velocity potential at instant t, at a point P in the purely supersonic region of the surface of a three-dimensional oscillating wing (Fig. 1) may be computed by

$$\Phi(x, y, \pm 0, t) = -\frac{U}{\pi} \exp(i\nu t) \int_{\xi_1}^{x} \exp\left[-i\nu \frac{U}{\beta^2 a^2}(x - \xi)\right] d\xi$$
$$\times \int_{y-(x-\xi)/\beta}^{y+(x-\xi)/\beta} \begin{Bmatrix} \Lambda_T(\xi, \eta) \\ \Lambda_B(\xi, \eta) \end{Bmatrix} \frac{\cos\{(\nu/\beta^2 a)[(x - \xi)^2 - \beta^2(y - \eta)^2]^{1/2}\}}{[(x - \xi)^2 - \beta^2(y - \eta)^2]^{1/2}}\, d\eta, \tag{9a}$$

where $z = \pm 0$ refers to the $\begin{Bmatrix}\text{top} \\ \text{bottom}\end{Bmatrix}$ surface of the wing.

A mixed supersonic region may be converted into a "psuedo-purely supersonic region" by Evvard's procedure of inserting a diaphragm into Region III of Fig. 1 which is an extension of the wing having the following properties:

a) It does not change the flow over the wing

b) It sustains no lift

With this supposition, the top and bottom surfaces of the wing may again be considered to be independent so that Eq. 9a applies; however the diaphragm slope is in general an unknown function which must be determined.

A part of the wing in Fig. 1 is shown enlarged in Fig. 4. In order to compute the

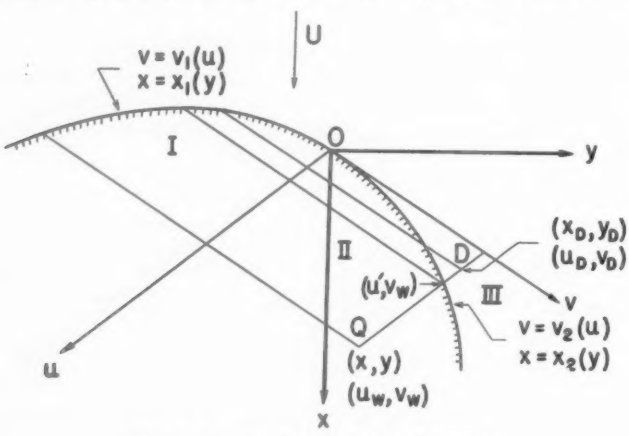

FIG. 4. A portion of the wing in Figure 1

velocity potential at the point Q, it is convenient to first consider a point D located on the trace of the upstream facing Mach cone from Q, in the diaphragm plane. Let the unknown downwash and the effective slope of the streamline on the top surface of the

diaphragm be $w_{DT}(x_D, y_D)$ and $\Lambda_{DT}(x_D, y_D)$ respectively. Then, by Eqs. 9 and 17, it is found that

$$\begin{aligned}\Phi_{DT}(x_D, y_D, +0, t) &= -\frac{1}{\pi}\exp(i\nu t)\\ &\times \iint_{S_W} w_T(\xi,\eta)\frac{\cos\{(\nu/\beta^2 a)[(x_D-\xi)^2-\beta^2(y_D-\eta)^2]^{1/2}\}}{[(x_D-\xi)^2-\beta^2(y_D-\eta)^2]^{1/2}\exp[i\nu(U/\beta^2a^2)(x_D-\xi)]}\,d\xi\,d\eta\\ &-\frac{1}{\pi}\exp(i\nu t)\\ &\times \iint_{S_D} w_{DT}(\xi,\eta)\frac{\cos\{(\nu/\beta^2 a)[(x_D-\xi)^2-\beta^2(y_D-\eta)^2]^{1/2}\}}{[(x_D-\xi)^2-\beta^2(y_D-\eta)^2]^{1/2}\exp[i\nu(U/\beta^2a^2)(x_D-\xi)]}\,d\xi\,d\eta,\end{aligned} \tag{18}$$

where S_W is the region of the wing and S_D is the region of the diaphragm included in the upstream facing Mach cone from $D(x_D, y_D, +0)$, at instant t. The regions of integration S_W and S_D are most easily expressed in terms of the oblique u, v coordinates defined as follows:

$$u = \frac{M}{2\beta}(\xi-\beta\eta), \qquad \xi = \frac{\beta}{M}(v+u),$$
$$\text{or} \tag{19}$$
$$v = \frac{M}{2\beta}(\xi+\beta\eta), \qquad \eta = \frac{1}{M}(v-u),$$

With these coordinate transformations, the point (x_D, y_D) is transformed into (u_D, v_D), where

$$u_D = \frac{M}{2\beta}(x_D-\beta y_D) \qquad x_D = \frac{\beta}{M}(v_D+u_D)$$
$$\text{o} \tag{20}$$
$$v_D = \frac{M}{2\beta}(x_D+\beta y_D) \qquad y_D = \frac{1}{M}(v_D-u_D)$$

The surface integral Φ_{DT} in Eq. 18 will now be integrated in the u, v plane. Then, Eq. 18 becomes

$$\begin{aligned}\Phi_{DT}(u_D, v_D, +0, t) &= -\frac{1}{\pi M}\exp(i\nu t)\int_0^{u_D}\frac{du}{(u_D-u)^{1/2}\exp[(i\nu/\beta a)(u_D-u)]}\\ &\times\int_{v_1(u)}^{v_2(u)}\frac{W_T(u,v)\cos\{(2\nu/M\beta a)[(u_D-u)(v_D-v)]^{1/2}\}}{(v_D-v)^{1/2}\exp[(i\nu/\beta a)(v_D-v)]}\,dv\\ &-\frac{1}{\pi M}\exp(i\nu t)\int_0^{u_D}\frac{du}{(u_D-u)^{1/2}\exp[(i\nu/\beta a)(u_D-u)]}\\ &\times\int_{v_2(u)}^{v_D}\frac{W_{DT}(u,v)\cos\{(2\nu/M\beta a)[(u_D-u)(v_D-v)]^{1/2}\}}{(v_D-v)^{1/2}\exp[(i\nu/\beta a)(v_D-v)]}\,dv.\end{aligned} \tag{21}$$

where $w_T(u, v)$ is the downwash on the top surface of the wing, $w_{DT}(u, v)$ is the down-

wash on the top surface of the diaphragm, the area bounded by $0 \le u \le u_D$ and $v_1(u) \le v \le v_2(u)$ is S_W, and the area bounded by $0 \le u \le u_D$ and $v_2(u) \le v \le v_D$ is S_D (Fig. 4).

Similarly, for the corresponding point $D(u_D, v_D, -0, t)$ on the bottom surface of the diaphragm, it is seen that

$$\begin{aligned}\Phi_{DB}(u_D, v_D, -0, t) &= \frac{1}{\pi M} \exp(i\nu t) \int_0^{u_D} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_1(u)}^{v_2(u)} \frac{w_B(u, v) \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} dv \\ &+ \frac{1}{\pi M} \exp(i\nu t) \int_0^{u_D} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_2(u)}^{v_D} \frac{w_{DB}(u, v) \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} dv.\end{aligned} \tag{22}$$

Off the wing, the downwash must be continuous. In terms of the effective slopes of the stream lines, this condition is, with the sign convention of Fig. 2,

$$\Lambda_{DT}(u, v) = -\Lambda_{DB}(u, v) = \Lambda_D(u, v). \tag{23}$$

From Eq. 3 it is found that in Region III

$$\Phi_{DT}(x, y, +0, t) = \Phi_{DB}(x, y, -0, t) + F(x - Ut, y), \tag{24}$$

where F is an integration function. The foremost Mach cone (Fig. 4) from the origin, 0, represents a line of infinitesimal disturbance along which $F(x - Ut, y)$ can be set equal to zero at all times. F remains zero along y = constant lines for values of x not intercepted by the wing (Ref. 1). Therefore, in Eq. 24, F may be put to zero. Behind a trailing edge, F may be different from zero and the theory must be modified. Then, from Eqs. 21, 22, 23 and 24 (with $F = 0$), it is seen that

$$\begin{aligned}&\frac{1}{2} \int_0^{u_D} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u, v) - \Lambda_T(u, v)] \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} dv \\ &= \int_0^{u_D} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_2(u)}^{v_D} \frac{\Lambda_D(u, v) \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} dv.\end{aligned} \tag{25}$$

When $\nu = 0$, this reduces to

$$\begin{aligned}&\frac{1}{2} \int_0^{u_D} \frac{du}{(u_D - u)^{1/2}} \int_{v_1(u)}^{v_2(u)} \frac{\Lambda_B(u, v) - \Lambda_T(u, v)}{(v_D - v)^{1/2}} dv \\ &= \int_0^{u_D} \frac{du}{(u_D - u)^{1/2}} \int_{v_2(u)}^{v_D} \frac{\Lambda_D(u, v)}{(v_D - v)^{1/2}} dv.\end{aligned} \tag{26}$$

Inasmuch as the limits of integration of the u-integrals are the same for *all values* of u_D and the integrals are "faltung" integrals, the two integrals with respect to v may be equated along lines of *constant* u that extends across the wing and the diaphragm (Fig. 4). Therefore,

$$\int_{v_2(u)}^{v_D} \frac{\Lambda_D(u, v)}{(v_D - v)^{1/2}} \, dv = \int_{v_1(u)}^{v_2(u)} \frac{\Lambda_B(u, v) - \Lambda_T(u, v)}{2(v_D - v)^{1/2}} \, dv. \tag{27}$$

This is the fundamental result of Ref. 2 and also is the basic equation of Ref. 3. The above argument is valid because the terms containing u_D do not appear in the v-integrals, and hence Eq. 27 is true for all u_D's on the line $v = v_D$.

The parallel treatment of Eq. 25 would be possible if the terms containing $(u_D - u)$ $(v_D - v)$ can be separated as in Eq. 26, under the integral signs. The present treatment represents a first attempt towards this end. The isolation of terms containing $(u_D - u)$ from terms containing $(v_D - v)$ such that the v-integral is free of the $(u_D - u)$ factor, may be accomplished by the following procedures.

The term $\{(v_D - v)\ (u_D - u)\}^{1/2}$ vanishes at (u_D, v_D), therefore Eq. 25 actually should be

$$\begin{aligned} &\lim_{\epsilon \to 0} \int_0^{u_D - \epsilon} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u, v) - \Lambda_T(u, v)] \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{2(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} \, dv \\ &= \lim_{\substack{\epsilon \to 0 \\ \epsilon_1 \to 0}} \int_0^{u_D - \epsilon} \frac{du}{(u_D - u)^{1/2} \exp[(i\nu/\beta a)(u_D - u)]} \\ &\times \int_{v_2(u)}^{v_D - \epsilon_1} \frac{\Lambda_D(u, v) \cos\{(2\nu/M\beta a)[(u_D - u)(v_D - v)]^{1/2}\}}{(v_D - v)^{1/2} \exp[(i\nu/\beta a)(v_D - v)]} \, dv. \end{aligned} \tag{25a}$$

The nature of the functions Λ_B, Λ_T and Λ_D must be such as to insure the existence of the improper integrals. Thus, except for the singularity (u_D, v_D); in the finite integration regions, the integrands are defined and bounded everywhere. Now, the circular functions are defined by power series; in particular, the power series expansion of the cosine function is

$$\cos z = \sum_{n=0}^{\infty} \frac{(-)^n}{(2n)!} z^{2n}. \tag{28}$$

The series (28) has the following properties (Ref. 8):

(1) It converges absolutely for all values of z (real and complex),
(2) It converges uniformly in any bounded domain of values of z, and consequently,
(3) It is a continuous function of z for all values of z.

Because of the uniform continuity, the cosine function in Eq. 25a may be expanded in an infinite series and the orders of integration and summation may be inverted. Thus

$$\sum_{n=0}^{\infty} \frac{(-)^n(\nu/\beta a)^{2n}(1/M)^{2n}\pi^{1/2}}{n!\Gamma(n+1/2)}$$
$$\times \lim_{\epsilon\to 0}\int_0^{u_D-\epsilon} \frac{(u_D-u)^{n-1/2}\,du}{\exp\,[(i\nu/\beta a)(u_D-u)]}\int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_D-v)^{n-1/2}}{2\exp\,[(i\nu/\beta a)(v_D-v)]}\,dv$$
$$= \sum_{n=0}^{\infty} \frac{(-)^n(\nu/\beta a)^{2n}(1/M)^{2n}\pi^{1/2}}{n!\Gamma(n+1/2)} \tag{25b}$$
$$\times \lim_{\epsilon\to 0}\int_0^{u_D-\epsilon} \frac{(u_D-u)^{n-1/2}\,du}{\exp\,[(i\nu/\beta a)(u_D-u)]}\lim_{\epsilon_1\to 0}\int_{v_2(u)}^{v_D-\epsilon_1} \frac{\Lambda_D(u,v)(v_D-v)^{n-1/2}}{\exp\,[(i\nu/\beta a)(v_D-v)]}\,dv.$$

With the conviction that the improper integrals under question exist, the "lim" signs may be left out.

In Eq. 25b, unlike in Eq. 25a, the v-integrals do not contain u_D terms, and the problem has been reduced to one analogous to that of Eq. 26. Now, it may be pointed out that since Eq. 25a is derived by equating the velocity potential on the top and bottom surfaces of the diaphragm in Region III (Fig. 4), the two sides of Eq. 25b may be conveniently considered as power series in $(1/M)$ of a potential function Φ, satisfying the original linear differential equation, Eq. 1; consequently corresponding terms may be equated.

Therefore, for *constant* value of v_D, with n being any positive integer,

$$\int_{v_2(u)}^{v_D} \frac{\Lambda_D(u,v)(v_D-v)^{n-1/2}}{\exp\,[(i\nu/\beta a)(v_D-v)]}\,dv = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_D-v)^{n-1/2}}{2\exp\,[(i\nu/\beta a)(v_D-v)]}\,dv. \tag{29}$$

In this system of simultaneous integral equations $\Lambda_B(u, v)$ and $\Lambda_T(u, v)$ are known while $\Lambda_D(u, v)$ is unknown. Consider, say, $(N + 1)$ integral equations corresponding to $n = 0, 1, 2, \cdots N$. (Of course, in the limit, $N \to \infty$). In order that these $(N + 1)$ simultaneous equations may determine one unknown Λ_D, it is necessary that the $(N + 1)$ equations are not mutually independent, that is, the $(N + 1)$ equations are reducible to *one* equation. In fact, this is true for the given system. For instance, when $n = 1$, it is obtained from Eq. 29 that

$$\int_{v_2(u)}^{v_D} \frac{\Lambda_D(u,v)(v_D-v)^{1/2}\,dv}{\exp\,[(i\nu/\beta a)(v_D-v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_D-v)^{1/2}\,dv}{2\exp\,[(i\nu/\beta a)(v_D-v)]}. \tag{30}$$

Carry out a differentiation of Eq. 30 with respect to v_D. The result of this differentiation plus $(i\nu/\beta a)$ times Eq. 30 yields

$$\int_{v_2(u)}^{v_D} \frac{\Lambda_D(u,v)(v_D-v)^{-1/2}\,dv}{\exp\,[(i\nu/\beta a)(v_D-v)]} = \int_{v_1(u)}^{v_2(u)} \frac{[\Lambda_B(u,v)-\Lambda_T(u,v)](v_D-v)^{-1/2}\,dv}{2\exp\,[(i\nu/\beta a)(v_D-v)]} \tag{31}$$

which is Eq. 29 for $n = 0$. Therefore, when Λ_D satisfies Eq. 30, it also satisfies Eq. 31. This argument can be carried on, by induction, to include the case for every n. Therefore, the system given by Eq. 29 is consistent and determines an unique function Λ_D.

For the determination of the contribution of the diaphragm on the velocity potential at a point $Q(u_W, v_W, \pm 0)$ on the top or bottom surface of the wing, it is not necessary

to solve the integral equation, Eq. 29, explicitly. Let this contribution be called $\Phi_{W_D}(u_W, v_W, \pm 0, t)$ (see Fig. 4). Then,

$$\begin{aligned}\Phi_{W_D}(u_W, v_W, \pm 0, t) &= \mp\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_W - u)^{1/2}\exp[(i\nu/\beta a)(u_W - u)]}\\ &\times\int_{v_2(u)}^{v_W}\frac{\Lambda_D(u,v)\cos\{(2\nu/M\beta a)[(u_W - u)(v_W - v)]^{1/2}\}\,dv}{(v_W - v)^{1/2}\exp[(i\nu/\beta a)(v_W - v)]}\\ &= \mp\frac{U}{\pi M}\exp(i\nu t)\sum_{n=0}^{\infty}\frac{(-)^n(\nu/\beta a)^{2n}(1/M)^{2n}\pi^{1/2}}{n!\Gamma(n+1/2)}\\ &\times\int_0^{u'}\frac{(u_W - u)^{n-1/2}\,du}{\exp[(i\nu/\beta a)(u_W - u)]}\int_{v_2(u)}^{v_W}\frac{\Lambda_D(u,v)(v_W - v)^{n-1/2}\,dv}{\exp[(i\nu/\beta a)(v_W - v)]},\end{aligned}\tag{32}$$

where u' is the u-coordinate of the intersection point of the curves: $v = v_2(u)$ and $v = v_W$, i.e. $v_2(u') = v_W$.

By comparing

$$\int_{v_2(u)}^{v_W}\frac{\Lambda_D(u,v)(v_W - v)^{n-1/2}}{\exp[(i\nu/\beta a)(v_W - v)]}\,dv \qquad\text{with}\qquad \int_{v_2(u)}^{v_D}\frac{\Lambda_D(u,v)(v_D - v)^{n-1/2}}{\exp[(i\nu/\beta a)(v_D - v)]}\,dv,$$

it is seen that they are identical if every v_D in the latter is replaced by v_W. But the value of v_D along the v = constant line passing through the point $(u_W, v_W, \pm 0)$ is v_W (Fig. 4). Hence v_D may be replaced by v_W in Eq. 29 and Eq. 32 becomes

$$\begin{aligned}\Phi_{W_D}(u_W, v_W, \pm 0, t) &= \mp\frac{U}{\pi M}\exp(i\nu t)\sum_{n=0}^{\infty}\frac{(-)^n(\nu/\beta a)^{2n}(1/M)^{2n}\pi^{1/2}}{n!\Gamma(n+1/2)}\\ &\times\int_0^{u'}\frac{(u_W - u)^{n-1/2}\,du}{\exp[(i\nu/\beta a)(u_W - u)]}\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v) - \Lambda_T(u,v)](v_W - v)^{n-1/2}}{2\exp[(i\nu/\beta a)(v_W - v)]}\,dv\\ &= \mp\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_W - u)^{1/2}\exp[(i\nu/\beta a)(u_W - u)]}\\ &\times\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v) - \Lambda_T(u,v)]\cos\{(2\nu/M\beta a)[(u_W - u)(v_W - v)]^{1/2}\}}{2(v_W - v)^{1/2}\exp[(i\nu/\beta a)(v_W - v)]}\,dv.\end{aligned}\tag{33}$$

In Eq. 33 an important theorem is established. The theorem may be stated as follows:

Theorem 3. In the computation of the velocity potential at an instant t at a point Q in the mixed supersonic region of an oscillating wing at supersonic speed, the contribution of the diaphragm may be evaluated by Eq. 33. In other words, the contribution of the diaphragm can be evaluated by an equivalent integration over a portion of the wing surface. Now, the velocity potential Φ at point Q on the top wing surface at instant t may be computed. It is (Fig. 4)

$$
\begin{aligned}
\Phi(u_W, v_W, +0, t) &= -\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_W-u)^{1/2}\exp[(i\nu/\beta a)(u_W-u)]} \\
&\quad\times\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v)-\Lambda_T(u,v)]\cos\{(2\nu/M\beta a)[(u_W-u)(v_W-v)]^{1/2}\}}{2(v_W-v)^{1/2}\exp[(i\nu/\beta a)(v_W-v)]}\,dv \\
&\quad-\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_W-u)^{1/2}\exp[(i\nu/\beta a)(u_W-u)]} \\
&\quad\times\int_{v_1(u)}^{v_2(u)}\frac{\Lambda_T(u,v)\cos\{(2\nu/M\beta a)[(u_W-u)(v_W-v)]^{1/2}\}}{(v_W-v)^{1/2}\exp[(i\nu/\beta a)(v_W-v)]}\,dv \\
&\quad-\frac{U}{\pi M}\exp(i\nu t)\int_{u'}^{u_W}\frac{du}{(u_W-u)^{1/2}\exp[(i\nu/\beta a)(u_W-u)]} \\
&\quad\times\int_{v_1(u)}^{v_W}\frac{\Lambda_T(u,v)\cos\{(2\nu/M\beta a)[(u_W-u)(v_W-v)]^{1/2}\}}{(v_W-v)^{1/2}\exp[(i\nu/\beta a)(v_W-v)]}\,dv.
\end{aligned}
\tag{34}
$$

In Eq. 34 the first surface integral represents the contribution from the diaphragm, while the last two surface integrals are the contribution from the top surface of the wing. By combining the first and second surface integrals, it is seen that

$$
\begin{aligned}
\Phi(u_W, v_W, +0, t) &= -\frac{U}{\pi M}\exp(i\nu t)\int_0^{u'}\frac{du}{(u_W-u)^{1/2}\exp[(i\nu/\beta a)(u_W-u)]} \\
&\quad\times\int_{v_1(u)}^{v_2(u)}\frac{[\Lambda_B(u,v)+\Lambda_T(u,v)]\cos\{(2\nu/M\beta a)[(u_W-u)(v_W-v)]^{1/2}\}}{2(v_W-v)^{1/2}\exp[(i\nu/\beta a)(v_W-v)]}\,dv \\
&\quad-\frac{U}{\pi M}\exp(i\nu t)\int_{u'}^{u_W}\frac{du}{(u_W-u)^{1/2}\exp[(i\nu/\beta a)(u_W-u)]} \\
&\quad\times\int_{v_1(u)}^{v_W}\frac{\Lambda_T(u,v)\cos\{(2\nu/M\beta a)[(u_W-u)(v_W-v)]^{1/2}\}}{(v_W-v)^{1/2}\exp[(i\nu/\beta a)(v_W-v)]}\,dv.
\end{aligned}
\tag{35}
$$

Eq. 35 may be restated in the following theorem:

Theorem 4. The velocity potential, in the mixed supersonic region on the top surface of a three-dimensional oscillating wing, may be computed by Eq. 35 or, in the x, y coordinates,

$$
\begin{aligned}
\Phi(x, y, +0, t) &= -\frac{U}{\pi}\exp(i\nu t) \\
&\quad\times\iint_{S_{W_1}}\frac{[\Lambda_B(\xi,\eta)+\Lambda_T(\xi,\eta)]\cos\{(\nu/\beta^2 a)[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}\}}{2[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}\exp[(i\nu U/\beta^2 a^2)(x-\xi)]}\,d\eta\,d\xi \\
&\quad-\frac{U}{\pi}\exp(i\nu t)\iint_{S_{W_2}}\frac{\Lambda_T(\xi,\eta)\cos\{(\nu/\beta^2 a)[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}\}\,d\eta\,d\xi}{[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}\exp[(i\nu U/\beta^2 a^2)(x-\xi)]},
\end{aligned}
\tag{35a}
$$

where S_{W_1} is the area bounded by $0 \leq u \leq u'$ and $v_1(u) \leq v \leq v_2(u)$ and S_{W_2} is the area bounded by $u' \leq u \leq u_W$ and $v_1(u) \leq v \leq v_W$.

The corresponding result for a point on the bottom surface of the wing may be obtained by interchanging Λ_T and Λ_B. It may be noted that for a wing of zero thickness, $\Lambda_B(\xi, \eta) + \Lambda_T(\xi, \eta) = 0$ so the integrals over S_{W_1} vanish. The simple "equivalent area" theorem established by Evvard in Ref. 2 for steady flows is thus seen to be valid for oscillating flows.

Discussion. (A) In Ref. 4, the boundary value problem for the determination of the velocity potential in the purely supersonic region of a wing in unsteady motion at supersonic speed was treated by source-superposition method in a quite general manner. In fact, theorems 1 and 2 mentioned above are included in Garrick and Rubinow's results. On specializing to considerations of an oscillating wing, the derivation of Eq. 16 becomes very simple. The derivation of the same equation in Ref. 4 is more complicated.

(B) Since an arbitrary downwash function can usually be expanded as a Fourier series, a harmonically oscillatory motion may be considered as a basis for building up more general motion for a nonstationary wing. This paves the way to construct a proof for a theorem applicable to more general nonstationary wings. Evvard, in Ref. 1, treated

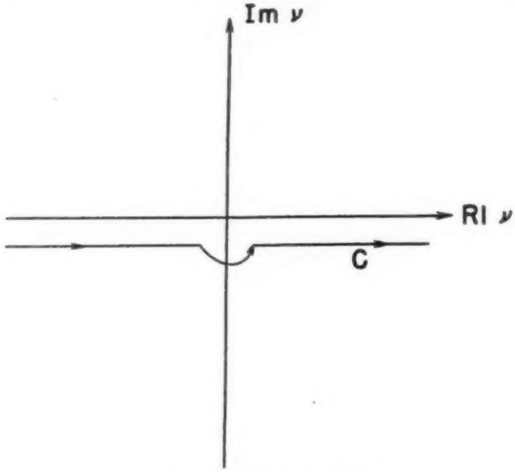

FIG. 5. Contour C in ν-plane.

the general mixed boundary value problem by the source superposition method. His results include the present theorems 3 and 4; however, his analysis was not carried through to the present point.

(C) A particular type of motion which is of both theoretical and practical interest and which demonstrates simply that theorems 3 and 4 apparently do not apply in the simple equivalent area form to all nonsteady motions, is the so-called "unit step" motion, in which a wing at rest starts abruptly at a certain instant and then maintains a steady motion. For composition of the velocity potential for a wing with motion of this nature, the "unit step" source will be useful. The "unit step" source can be derived from an oscillating source by a contour integration in the ν plane,

$$\phi_2(x, y, z, t) = \frac{1}{2\pi i}\int_C \varphi_1 \frac{d\nu}{\nu} = \frac{\beta}{2\pi i r_1}\int_C \cos\left(\frac{\nu r_1}{\beta^2 a}\right) \exp\left\{i\nu\left[t - \frac{U}{\beta^2 a^2}(x - \xi)\right]\right\}\frac{d\nu}{\nu}, \tag{36}$$

where C is the contour shown in Fig. 5. By writing the cosine term in exponential form Eq. 36 can be shown to yield

$$\phi_2(x, y, z, t) = \frac{\beta}{2r_1}\left[H\left(t - \frac{Ux}{\beta^2 a^2} + \frac{r_1}{\beta^2 a}\right) + H\left(t - \frac{Ux}{\beta^2 a^2} - \frac{r_1}{\beta^2 a}\right)\right], \tag{36a}$$

where $H(\mu)$ is the "unit step" function having the properly that

$$H(\mu) = \begin{cases} 1 & \mu > 0, \\ 0 & \mu < 0. \end{cases} \tag{37}$$

Now, draw a sphere of radius (at) enclosed in the circular cone from the "unit step" source at (ξ, η, ζ), with the center of the sphere located at a distance (Ut) from (ξ, η, ζ)

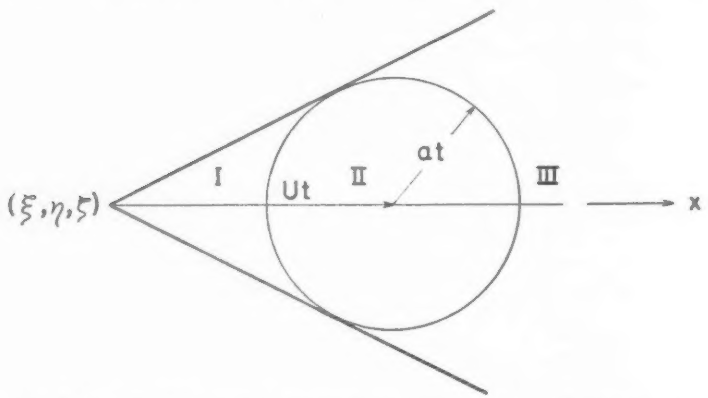

FIG. 6. Region of influence of the "unit step" source at (ξ, η, ζ) at instant t.

(Fig. 6). Then the region of influence of the source is divided into *three* regions (by Eq. 37),

(1) In region I, the influence is equivalent to that of a steady source.
(2) In region II, the influence is equivalent to that of a steady source of half strength.
(3) In region III, no influence of the source will be felt.

The region of dependence for a point (x, y, z) will consist of three similar regions.

Consider the lift problem of a rectangular flat plate wing performing a "unit step" motion. Suppose that the velocity potential at a point S in the mixed supersonic region near the wing tip is to be computed at an instant t_1, such that $at_1 < | y |$ (Fig. 7). According to the above argument, the condition at S will depend on both regions A and B and the wing tip will have no influence. But in accordance with equivalent area form of Theorem 4 the domain of dependence at S would exclude the shaded region in Fig. 7 in the computation of the velocity potential at S, at instant t_1. This provides an example

of the fallacy of the equivalent area interpretation of Theorem 4 for arbitrary nonsteady motions.

Therefore, theorems 3 and 4 are *not* directly applicable to "unit step" wings. This fact is indicated (but not proved) by Eq. 36, because the operation of the contour integration will carry the cosine function to infinity such that the argument of Eq. 25b breaks down in the proof of Theorem 3.

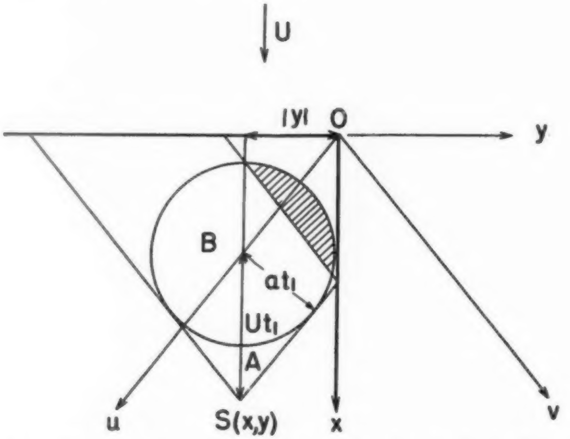

FIG. 7. The wing tip region of a rectangular flat plate performing "unit step" motion at instant t_1.

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THE OSCILLATING RECTANGULAR AIRFOIL AT SUPERSONIC SPEEDS*

BY
JOHN W. MILES**
University of California, Los Angeles

Summary. The pressure distribution on a quarter infinite, zero thickness airfoil having a prescribed distribution of downwash (on the wing only), which exhibits a harmonic time dependence, is determined by a Fourier transform solution of the linearized, potential equation for supersonic flow. The solution is effected with the aid of the Wiener-Hopf technique and leads to a Green's function, which may be expressed either as a finite, definite integral or as an expansion in powers of a dimensionless frequency parameter. It is shown that the results are applicable to the calculation of the forces and moments on rectangular airfoils of effective aspect ratio ($A \cot \theta$, where θ is the Mach angle) greater than unity. It appears that the force and moment coefficients of practical interest may be expressed in terms of known functions, including certain integrals which have been calculated for the two-dimensional, oscillating airfoil. The extension of the two-dimensional results to rectangular wings for which the prescribed downwash is constant along the span is particularly simple. The extension of the results for harmonic time dependence to the step function (Heaviside) case is indicated.

1. Introduction. The linearized, two-dimensional problem of the oscillating airfoil at supersonic speeds has been studied by a number of analysts, using a variety of approaches. This work has recently been collected and summarized in two reports prepared by Biot et. al.,[1,2] which give both the methods of analysis[1] and the numerical results.[2] (The reports also include the subsonic, compressible case.) These results are probably adequate for the strip theory analysis of wings with supersonic leading and trailing edges and aspect ratios sufficiently high to render tip effects small. (For an approximate treatment of the swept wing, reference may be made to a recent paper by the writer.[3]) Unfortunately, those wings which meet the limitation of supersonic leading and trailing edges are generally characterized by small aspect ratio; moreover, the most serious supersonic flutter problems indicated by two dimensional analyses of such wings frequently occur in the near sonic regime, where tip effects are by no means negligible. Accordingly, it is of considerable practical importance to consider the three dimensional problem of the oscillating airfoil.

The problem selected for study in the present paper is that of the rectangular wing tip, since it is the simplest three dimensional configuration (excepting those wings with no subsonic edges) of practical import. The results may be applied directly to the rectangular airfoil of aspect ratio sufficiently large to prevent the Mach waves from the leading edge wing tips from intersecting one another forward of the trailing edge and,

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*Received November 21, 1949.

**Consultant, U.S. Naval Ordnance Test Station, Inyokern, Calif. The author wishes to thank Dr. D. E. Zilmer for his careful criticism and helpful suggestions in the preparation of the manuscript.

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indirectly, to the case where these Mach waves intersect on the wing but do not intersect the opposite side edges. It is also possible to solve the case of arbitrarily small aspect ratio, but the results are so complex as to be of dubious practical value.

The method of solution to be used follows the Wiener-Hopf technique,[4] which has previously been applied to the rectangular wing in a steady flow.[5]

2. Statement of problem. A thin, rectangular airfoil is located in the vicinity of the plane $z = 0$, and its projection there occupies the first quadrant of the xy plane. A flow of supersonic velocity U is directed along the positive x axis, so that the leading edge of the airfoil is projected on the positive y axis and the port side edge on the positive x axis, as shown in Fig. 1. The boundary conditions are linearized in the usual manner, so that they may be applied at the projection of the airfoil on the plane $z = 0$, rather

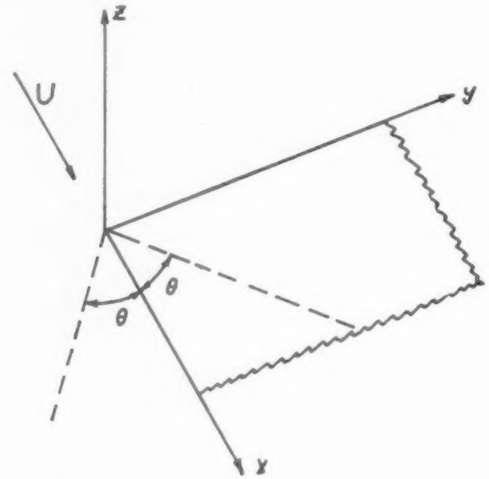

FIG. 1. x, y, z axes and projection of airfoil on xy plane.

than at the airfoil proper. The equations of flow will also be linearized,[6] so that the problem may be subdivided into antisymmetric and symmetric cases with respect to the plane $z = 0$; only the former case is of interest here, since the latter situation does not give rise to lift. Accordingly, it is sufficient to consider the half space $z > 0$ and to apply the boundary conditions appropriate to the airfoil in the plane $z = 0+$. The problem to be solved is then the specification of the perturbation pressure over the first quadrant of the plane $z = 0+$ from a knowledge of the downwash there.

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[4]N. Wiener and R. Paley, *The Fourier transform in the complex domain*, Amer. Ma. Soc. Colloq. Publ. **19**, Ch. IV (1934); E. C. Titchmarsh, *Theory of Fourier integrals*, Oxford Press (1937) 339-349; E. Reissner, J. Ma. & Ph. **20**, 219-223 (1941).

[5]J. W. Miles, *On the rectangular airfoil at supersonic speeds*, No. Amer. Avia. Report AL 866 (1948).

[6]A more complete discussion of the linearizing process and its various aspects is given by P. A. Lagerstrom, *Linearized supersonic theory of conical wings*, J.P.L. Progress Report 4-36. California Institute of Technology (1947); NACA T.N. 1685 (1948).

The vector, *perturbation* velocity ($\mathbf{q}$) due to the presence of the airfoil in the flow is specified as the gradient of a velocity potential $Ub\phi$, viz:

$$\mathbf{q}(x, y, z, t) = Ub\nabla\phi(x, y, z, t) \tag{2.1}$$

(b is a characteristic length, to be chosen in any convenient manner.) The gage pressure follows from Newton's law as:

$$p(x, y, z, t) = -\rho Ub \frac{D}{Dt}\phi(x, y, z, t) \tag{2.2}$$

where D/Dt is the time differentiation operator in a fixed reference frame and, in linearized form, is given by:

$$\frac{D}{Dt} = U\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \tag{2.3}$$

The condition of continuity, after linearization, leads to the scalar Helmholtz equation in the fixed reference frame, viz.:

$$\nabla^2\phi = \frac{1}{c^2}\frac{D^2}{Dt^2}\phi \tag{2.4}$$

where c is the sonic velocity for the ambient stream conditions.

At this point, it is convenient to introduce the harmonic time dependence exp ($i\omega t$), the Mach angle θ, the frequency parameter κ, the dimensionless coordinates (x', y', z'), and the modified potential, pressure, and downwash functions $\psi(x', y', z')$, $\gamma(x', y')$, and $\alpha(x', y')$, in accordance with the relations

$$\theta = \sin^{-1}(c/U) \tag{2.5}$$

$$\kappa = (\omega b/c)\tan\theta \tag{2.6}$$

$$x = (b\cot\theta)\,x' \tag{2.7a}$$

$$y = by' \tag{2.7b}$$

$$z = bz' \tag{2.7c}$$

$$\phi(x, y, z, t) = \exp[i(\omega t - \kappa x'\csc\theta)]\psi(x', y', z') \tag{2.8}$$

$$\frac{D}{Dt}\phi(x, y, 0+, t) = (U\tan\theta/b)\exp[i(\omega t - \kappa x'\csc\theta)]\gamma(x', y') \tag{2.9a}$$

$$\gamma(x', y') = \left(\frac{\partial}{\partial x'} - i\kappa\sin\theta\right)\psi(x', y', 0+) \tag{2.9b}$$

$$\alpha(x', y') = -\frac{\partial}{\partial z'}\psi(x', y', 0+) \tag{2.10}$$

Substitution of Eq. (2.8) in Eq. (2.4) yields the reduced equation

$$\psi_{y'y'} + \psi_{z'z'} = \psi_{x'x'} + \kappa^2\psi \tag{2.11}$$

while substituting Eq. (2.9a) in Eq. (2.2) yields

$$p(x, y, 0+, t) = -\rho U^2 \tan \theta \exp [i(\omega t - \kappa x' \csc \theta)]\gamma(x', y') \tag{2.12}$$

The boundary value problem to be solved now may be posed as: find a solution to Eq. (2.11) which satisfies the boundary conditions

$$\psi_{z'}(x', y', 0+) = -\alpha_+(x', y'), \qquad y' > 0 \tag{2.13}$$

$$\gamma(x', y') = 0, \qquad y' \leq 0 \tag{2.14}$$

$$\psi(x', y', z') \equiv 0, \qquad x' < z' \tag{2.15}$$

Eq. (2.13) states that the (modified) downwash $\alpha_+(x', y')$, is prescribed on the airfoil; Eq. (2.14) states that the pressure must vanish off the airfoil, since it is presumed asymmetric with respect to z, and only the airfoil is capable of supporting a discontinuity in pressure; and Eq. (2.15) states that the disturbance is propagated downstream and must vanish forward of the Mach waves originating at the leading edge of the airfoil.

3. Fourier integral formulation. A general solution to Eq. (2.11) may be conveniently formulated in terms of Fourier integrals. The Fourier transformation of a function of the space coordinates (x, y) into its representative in the (μ, ν) spectrum will be denoted by a transition from lower to upper case letters in the functional notation, and the conjugate transform operators T and $\overline{T}$ are defined by

$$f(x, y) = T\{F(\mu, \nu)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\nu F(\mu, \nu) \exp [i(\mu x + \nu y)] \tag{3.1a}$$

$$F(\mu, \nu) = \overline{T}\{f(x, y)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f(x, y) \exp [-i(\mu x + \nu y)] \tag{3.1b}$$

In general, the parameters μ and ν may be allowed complex values, but the paths of integration in the μ and ν planes must be suitably restricted in order to comply with both physical and mathematical requirements. Frequent reference will be made to Titchmarsh[7] and Campbell and Foster[8], simply by using the letters T and CF, followed by the appropriate equation number in the original source, although the notation used herein is not entirely consistent with either of these references.[9]

In addition to the entire transforms of Eq. (3.1), it is expedient to introduce the notation

$$F_+(\mu, \nu) = \overline{T}\{f(x, y)\mathbf{1}(x, y)\} \tag{3.2a}$$

$$F_-(\mu, \nu) = \overline{T}\{f(x, y)\mathbf{1}(x, -y)\} \tag{3.2b}$$

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[7]E. C. Titchmarsh, *Theory of Fourier integrals*, Oxford Univ. Press (1937).

[8]G. A. Campbell and R. M. Foster, *Fourier integrals for practical applications*, Bell Tel. Syst. Mono. B-584 (1942); also published by D. Van Nostrand and Co., New York, N. Y. (1948).

[9]In the case of transformation with respect to a single variable, we find it convenient, however, to use the notation of ref. 8, such that, e.g. $f(x) = T_\mu\{F(\mu)\} = 1/2\pi \int^{\infty}_{-\infty} d\mu\, F(\mu) \exp (i\mu x)$.

Thus, the inversions of section 5 correspond to ref. 8.

$$\begin{aligned} 1(x, y) &= 1, \qquad x > 0 \quad \text{and} \quad y > 0 \\ & 0, \qquad x < 0 \quad \text{or} \quad y < 0 \end{aligned} \tag{3.3}$$

Eq. (3.3) defines the Heaviside step function in two variables, but if only one variable is indicated, e. g., $1(x)$, the step is in that variable alone. $F_+(\mu, \nu)$ then represents the transform of a function which vanishes off the wing, while $F_-(\mu, \nu)$ represents the transform of a function which vanishes on the wing, recalling, cf. Eq. (2.15), that the solution vanishes identically forward of the wing.

It is readily shown that an elementary solution to Eq. (2.11) is given by

$$\psi_0(x, y, z) = \exp [i(\mu x + \nu y) - \lambda z] \tag{3.4}$$

$$\lambda = [\nu^2 - (\mu^2 - \kappa^2)]^{1/2} \tag{3.5}$$

the sign of the exponent λz being chosen to represent a disturbance which is bounded for large z. The primes have been dropped from the coordinates, but they are assumed to be the dimensionless (primed) coordinates of Eq. (2.7). By virtue of the linearization of the problem, these elementary solutions may be synthesized with the aid of the Fourier integral to form solutions capable of satisfying prescribed boundary conditions. In particular, the most general solution to Eq. (2.11) reducing to $\gamma(x, y)$, cf. Eqs. (2.9b) and (2.14), is given by

$$\psi(x, y, z) = T\left\{\frac{\Gamma_+(\mu, \nu) \exp(-\lambda z)}{i(\mu - \kappa \sin \theta)}\right\} \tag{3.6}$$

provided that the paths of integration in the μ and ν planes are suitably chosen.

In order to determine the appropriate paths in the complex transform planes, it is necessary to establish a domain of regularity for $\Gamma_+(\mu, \nu)$. If it is simply assumed that $\gamma(x, y)$ is bounded for large x (the behavior of γ in y affects the behavior of Γ in ν only), it follows from Eq. (3.1b) that $\Gamma_+(\mu, \nu)$ will be regular in $\text{Im}(\mu) < 0$, so that the μ integration may be carried out along a path in the lower half of the complex μ plane. More specifically, $\Gamma_+(\mu, \nu)$ will be found to have a simple pole at $\mu = +(\kappa \csc \theta + i\epsilon)$ if the pressure on the wing, cf. Eq. (2.9a) behaves as $\exp(-\epsilon x)$ for large x, where ϵ is a positive real constant, and will have a zero at $\mu = \kappa \sin \theta$, as may be verified *a posteriori*. (Due to the zero in Γ at $\mu = \kappa \sin \theta$, Ψ does not have a pole there.) In the present analysis, it suffices to take $\epsilon = 0$, insofar as $\text{Im}(\mu) < 0$. Accordingly, the integrand in Eq. (3.6) is regular in μ except for a simple pole on the real axis and the branch points in λ, the latter being designated as $\pm\mu_0(\nu)$, where, cf. Eq. (3.5),

$$\lambda = (\mu_0^2 - \mu^2)^{1/2} \tag{3.7}$$

$$\mu_0(\nu) = (\kappa^2 + \nu^2)^{1/2} \tag{3.8}$$

The location of these branch points of course depends on ν, but if the path in the ν plane is chosen such that

$$|\text{Im}(\nu)| < \kappa \tag{3.9}$$

they will always possess real parts, and their imaginary parts will always be less (in magnitude) than $|\text{Im}(\nu)|$. It follows that, if the branch cuts from $\pm\mu_0$ are both extended

to $+i\infty$, the integrand of Eq. (3.6) will be regular everywhere in the half plane bounded by

$$\text{Im}(\mu) < -\mid \text{Im}(\nu) \mid < -\mid \text{Im}(\mu_0) \mid \tag{3.10}$$

and the phase of λ will be $\pi/2$, 0, and $-\pi/2$ for Im $(\mu) = 0$ and Re $(\mu) > \mu_0$, $\mid \text{Re}(\mu) \mid < \mu_0$ and Re $(\mu) < -\mu_0$, respectively. An appropriate path of integration is then one satisfying the restriction (3.10), as illustrated in Fig. 2.

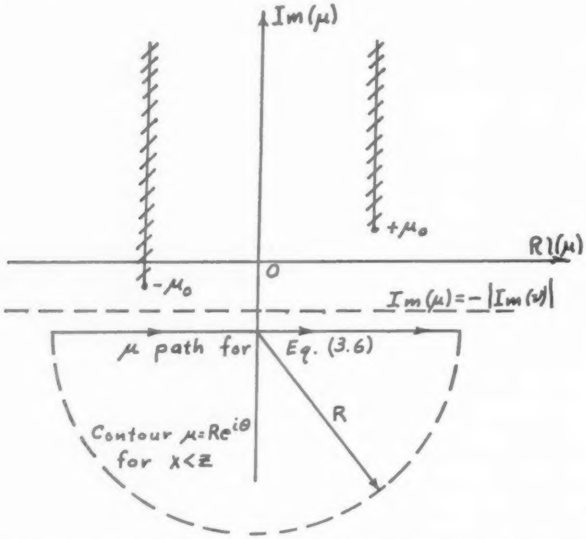

FIG. 2. μ plane, showing branch points $\pm\mu_0$ in λ, the branch cuts therefrom, the path of integration for Eq. (3.6), and the completion of this path for $x < z$.

Consider now the behavior of the solution (3.6) for $x < z$. The path of integration along the horizontal line chosen in accordance with Eq. (3.10) may be closed by a semi-circle in the lower half (μ) plane. If μ is denoted by $\text{Re}^{i\theta}$ along the latter path, the choice of phase for λ implies

$$\mid \exp(i\mu x - \lambda z) \mid \sim \mid \exp[i\mu(x - z)] \mid \sim \exp[-R(x - z)\sin\theta] \tag{3.11}$$

Since θ is negative along the semi-circle, the integrand of Eq. (3.6) is exponentially bounded, as long as $x < z$, and the contribution of this path to the contour integral therefore vanishes uniformly as R is allowed to become infinite (for $-\pi + \epsilon < \theta < -\epsilon$, $\epsilon > 0$; the portions $-\epsilon < \theta < 0$ and $-\pi < \theta < -\pi + \epsilon$ may be interpreted as belonging to the path associated with the transform path for ψ). Moreover, since the integrand is regular in the half plane bounded by Eq. (3.10), the entire contour integral vanishes by virtue of Cauchy's theorem, whence $\psi(x, y, z)$ vanishes identically for $x < z$, in satisfaction of Eq. (2.15).

The asymptotic behavior of the solution for large, positive x (i.e., the Trefftz plane) may be obtained by closing the contour in Im $(\mu) > 0$ plus "keyhole" paths around the

branch points $\pm\mu_0$, the only contributions to the integral coming from the latter paths and the pole at $\mu = +\kappa \csc \theta$.[10]

Turning to the ν plane, the branch points of λ may be designated as $\pm\nu_0$, where

$$\lambda = (\nu^2 - \nu_0^2)^{1/2} \tag{3.12}$$

$$\nu_0(\mu) = (\mu^2 - \kappa^2)^{1/2} \tag{3.13}$$

Considering $\nu_0(\mu)$ in the μ plane, the branch cuts are taken from $\pm\kappa$ to $\pm\infty$, so that the phase of ν_0 will be 0, $-\pi/2$, and $-\pi$ for Im $(\mu) = 0$ and Re $(\mu) > \kappa$, $|$ Re $(\mu)| < \kappa$,

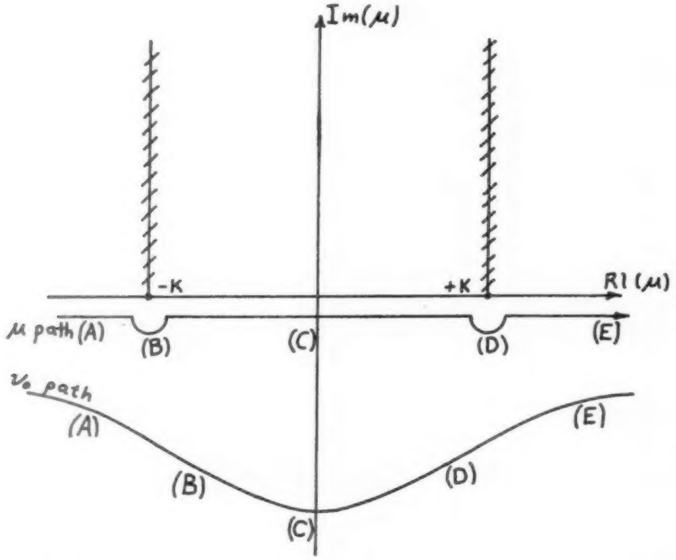

FIG. 3. μ plane, showing cuts from $\pm\kappa$ and paths of μ and $\nu_0(\mu)$. A, B, C, D, E, represent corresponding points on these paths.

and Re $(\mu) < -\kappa$, respectively. As μ follows the horizontal path designated by Eq. (3.10) $\nu_0(\mu)$ will follow a lower path, as shown in Fig. 3, such that

$$\text{Im } (\nu_0) < \text{Im } (\mu) \tag{3.14}$$

The cuts from the branch points $\pm\nu_0$ in the ν plane must be chosen such that Re $(\lambda) \geq 0$ (in order that the solution (3.6) will be bounded for all positive z) for all (μ, ν); accordingly, the path of integration in the ν plane must pass under one branch point and over

[10] If the path for Eq. (3.6) is allowed to approach the real axis (also taking $Im(\nu) = 0$) and is indented under $+\kappa \csc \theta$ and $\mp\mu_0$, the pole will give only half the contribution obtained for $Im(\mu) < 0$ and will also contribute an equal and opposite amount to the contour for $x < z$. It is necessary in this case to add an auxiliary solution, which is independent of x and is represented by $\pi\delta(\mu - \kappa \csc \theta)\Gamma_+(\kappa \csc \theta, \nu) \cdot \exp[-(\nu^2 - \kappa^2 \cot^2 \theta)^{1/2}z]$. See, e.g., J. W. Miles, *On linearized supersonic airfoil theory*, No. Amer. Avia., Inc. Report AL-801 (1948) pp. 15, 16, where the care of stationary flow ($\kappa = 0$) is discussed.

the other, and, in addition, the cuts must not interfer as Re (ν_0) changes sign, cf. Figs. 3. and 4. It follows that the cuts must run from $+\nu_0$ to $-i\infty$ and from $-\nu_0$ to $+i\infty$, since Im $(\nu_0) \leq 0$ for all μ, as shown in Fig. 4. The result of this choice is that the elementary solution $\psi_0(x, y, z)$, cf. Eq. (3.4), behaves as exp $[i\mu(x - z)]$ for large (absolute) values of μ and not too large values of ν, indicating the (Mach) wave front $x = z$, representing the locus of the waves originating at the leading edge of the wing.[11]

Returning to the solution (3.6), differentiating with respect to z, and substituting in Eq. (2.13) yields

$$\alpha_+(x, y) = T\{G(\mu, \nu)\Gamma_+(\mu, \nu)\}, \qquad y > 0 \tag{3.15}$$

$$G(\mu, \nu) = -i(\mu - \kappa \sin \theta)^{-1}[\nu^2 - \nu_0^2(\mu)]^{1/2} \tag{3.16}$$

It may be remarked that Eq. (3.15) is valid for all y if $\alpha_-(x, y)$ is added to $\alpha_+(x, y)$.

Eq. (3.15), for $y > 0$, and Eq. (2.14), for $y < 0$, together constitute a dual integral

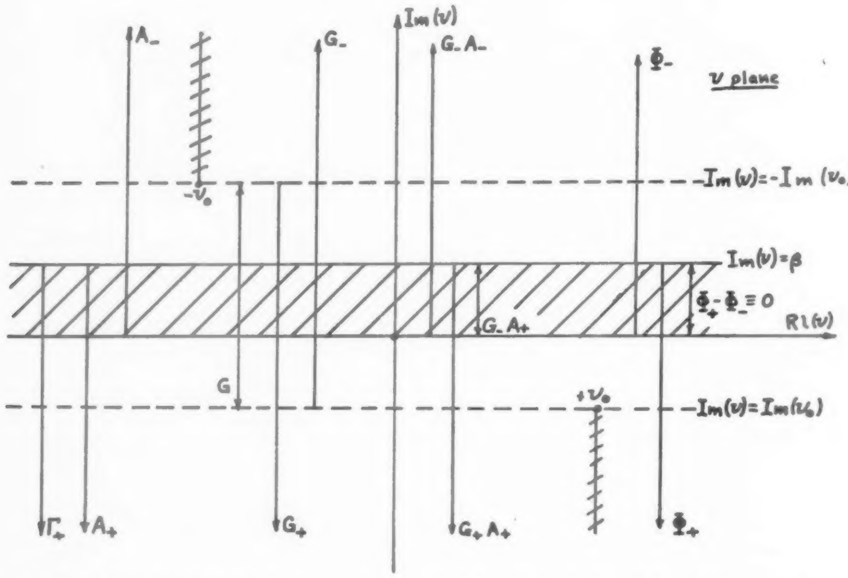

FIG. 4. ν plane, showing cuts from $\pm\nu_0$ and the domains of regularity of the various transforms.

equation (cf. T, pp. 334-342) for the determination of $\gamma(x, y)$. Its reduction to transform equations may be effected by taking the inverse transforms of Eqs. (3.15), *extended as valid for all* y, and (2.14). The results are

$$G(\mu, \nu)\Gamma_+(\mu, \nu) = A_+(\mu, \nu) + A_-(\mu, \nu) \tag{3.17a}$$

$$\Gamma_-(\mu, \nu) = 0 \tag{3.17b}$$

[11]See also Eq. (3.11) in this respect.

4. Solution of transform equation. The transform equation (3.17) may be solved with the aid of the Wiener-Hopf technique. While it is possible to treat an arbitrary distribution $\alpha_+(x, y)$, the difficulty of inverting the complicated transforms which arise in any general solution probably would render it impracticable. Accordingly, it will be assumed that

$$\alpha_+(x, y) = y^n \exp(-\beta y)1(y)\alpha_n(x) \tag{4.1}$$

which will be sufficiently general for most purposes.

The transform of Eq. (4.1) with respect to y is given by CF 524.2, while the transform with respect to x will be denoted by $A_n(\mu)$,[12] whence

$$A_+(\mu, \nu) = (2\pi)^{-1}\Gamma(n + 1)(\beta + i\nu)^{-n-1}A_n(\mu) \tag{4.2a}$$

$$A_n(\mu) = T_\mu\{\alpha_n(\xi)\} = \int_0^\infty d\xi \exp(-i\mu\xi)\alpha_n(\xi) \tag{4.2b}$$

Accordingly, $A_+(\mu, \nu)$ is regular in ν except for a pole of order $n + 1$ at $\nu = i\beta$. Moreover, it may be verified *a posteriori* that $\gamma(x, y)$ behaves as $\alpha_+(x, y)$ for large y and, therefore, also has a pole of order $n + 1$ at $\nu = i\beta$, while the fact that $\gamma(x, y)$ vanishes for $y = 0$, cf. Eq. (2.14), implies that its transform must vanish at least as $\nu^{-1-\epsilon}$, where ϵ is a positive constant, for large ν. Similarly, it may be verified that $\alpha_-(x, y)$ has a singularity of at worst y^{-1+} for small (negative) y and vanishes uniformly for large y, so that its transform vanishes for both large and small y. It follows that sufficient conditions for regularity of the various transforms in the complex μ and ν planes, and, therefore, suitably restricted domains in these planes, are

$$A_+(\mu, \nu): \quad \text{Im}(\mu) < -\beta, \quad \text{Im}(\nu) < \beta$$

$$A_-(\mu, \nu): \quad \text{Im}(\mu) < -\beta, \quad \text{Im}(\nu) > 0$$

$$\Gamma_+(\mu, \nu): \quad \text{Im}(\mu) < -\beta, \quad \text{Im}(\nu) < \beta$$

$$G(\mu, \nu): \quad \text{Im}(\mu) < -\beta, \quad |\,\text{Im}(\nu)\,| < |\,\text{Im}(\nu_0)\,| > \beta$$

The restriction $\text{Im}(\mu) < -\beta$ is in accordance with Eq. (3.10) and conveniently places the line $\text{Im}(\nu) = -\text{Im}(\nu_0)$ above $\text{Im}(\nu) = \beta$, as shown in Eq. 4. It should be remarked that these conditions refer only to the individual transforms and do not necessarily imply regularity of the complete transform in Eq. (3.6), for which it is sufficient to require Eqs. (3.9) and (3.10).

Since A_+ and Γ_+ are regular in the lower half ν plane and A_- in the upper half, it is expedient to split G into two functions, which are regular, respectively, in these two domains; thus:

$$G(\mu, \nu) = G_+(\mu, \nu)/G_-(\mu, \nu) \tag{4.3a}$$

$$G_+(\mu, \nu) = -i(\mu - \kappa \sin\theta)^{-1}(\nu + \nu_0)^{1/2}, \qquad \text{Im}(\nu) < -\text{Im}(\nu_0) \tag{4.3b}$$

$$G_-(\mu, \nu) = (\nu - \nu_0)^{-1/2}, \qquad \text{Im}(\nu) > \text{Im}(\nu_0) \tag{4.3c}$$

[12]See footnote 9 regarding notation.

It should be specifically noted that G_+ and G_- are not defined in accordance with Eq. (3.2).

The domains of regularity of A_+, A_-, Γ_+, G, G_+, and G_- in the ν plane are shown in Fig. 4. Moreover, if Eq. (3.17a) is multiplied through by $G_-(\mu, \nu)$, it is evident that

$$G_-(\mu, \nu)A_-(\mu, \nu) \quad \text{is regular in} \quad \mathrm{Im}\,(\nu) > 0$$

$$G_+(\mu, \nu)A_+(\mu, \nu) \quad \text{is regular in} \quad \mathrm{Im}\,(\nu) < \beta$$

$$G_-(\mu, \nu)A_+(\mu, \nu) \quad \text{is regular in} \quad 0 < \mathrm{Im}\,(\nu) < \beta$$

Since $A_+(\mu, \nu)$ is regular throughout the entire ν plane except for the pole of order $(n + 1)$ at $\nu = i\beta$, it is possible to remove the singular part of $G_-(\mu, \nu)A_+(\mu, \nu)$ to obtain a function which is regular in the upper half plane. Thus, introducing a Taylor expansion for $G_-(\mu, \nu)$ about $\nu = i\beta$,

$$\begin{aligned} G_-(\mu, \nu)A_+(\mu, \nu) &= \left[\sum_{m=0}^{n} \Gamma^{-1}(m + 1)G_-^{(m)}(\mu, i\beta)(\nu - i\beta)^m\right]A_+(\mu, \nu) \\ &+ \left[G_-(\mu, \nu) - \sum_{m=0}^{n} \Gamma^{-1}(m + 1)G_-^{(m)}(\mu, i\beta)(\nu - i\beta)^m\right]A_+(\mu, \nu) \end{aligned} \tag{4.4}$$

it is found that the second term on the right is $0(1)$ at $\nu = i\beta$, while the first term remains $0[(\nu - i\beta)^{-n-1}]$, and it follows that the two terms are regular in the upper and lower half ν planes, respectively. Multiplying Eq. (3.17a) through by $G_-(\mu, \nu)$ and subtracting from both sides the second term on the right of Eq. (4.4), the result may be written

$$\Phi_+(\nu) \equiv \Phi_-(\nu), \qquad 0 < \mathrm{Im}\,(\nu) < \beta \tag{4.5}$$

$$\Phi_+(\nu) = G_+(\mu, \nu)\Gamma_+(\mu, \nu) - \left[\sum_{m=0}^{n} \Gamma^{-1}(m + 1)G_-^{(m)}(\mu, i\beta)(\nu - i\beta)^m\right]A_+(\mu, \nu), \qquad \mathrm{Im}\,(\nu) < \beta \tag{4.6}$$

$$\begin{aligned} \Phi_-(\nu) = G_-(\mu, \nu)A_-(\mu, \nu) &+ \Bigg[G_-(\mu, \nu) \\ &- \sum_{m=0}^{n} \Gamma^{-1}(m + 1)G_-^{(m)}(\mu, i\beta)(\nu - i\beta)^m\Bigg]A_+(\mu, \nu), \qquad \mathrm{Im}\,(\nu) > 0 \end{aligned} \tag{4.7}$$

where $\Phi_+(\nu)$ and $\Phi_-(\nu)$ are regular in the indicated domains, are identical in the common strip of regularity $0 < \mathrm{Im}\,(\nu) < \beta$, and are, therefore, analytic continuations of the same function, say $\Phi(\nu)$, in the lower and upper half planes, respectively.

The problem is now reduced to the determination of the function $\Phi(\nu)$, and, since $\Phi(\nu)$ is regular for all ν, it must be an integral function.[13] Indeed, since it is regular at infinity as well as in the finite plane, it must be a constant, by virtue of Liouville's theorem.[14] To determine this constant, it suffices to examine $\Phi(\nu)$ for large ν. Recalling

[13]E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan Co., New York (1947), p. 105.
[14]Ibid.

that $\Gamma_+(\mu, \nu)$ must vanish at least as ν^{-1}, that $A_+(\mu, \nu)$ is of order ν^{-n-1}, and that $G_+(\mu, \nu)$ and $G_-(\mu, \nu)$ are $0(\nu^{1/2})$ and $0(\nu^{-1/2})$, respectively, it follows that $\Phi(\nu)$ is $0(\nu^{-1/2})$ and therefore vanishes identically.

Substituting the result $\Phi = 0$, G_+ and G_- from Eq. (4.3), A_+ from Eq. (4.2), and evaluating the derivative of $G_-(\mu, \nu)$, viz.

$$G_-^{(m)}(\mu, \nu) = (-)^m \pi^{-1/2} \Gamma(m + 1/2)(\nu - \nu_0)^{-m-1/2} \tag{4.8}$$

the results for Γ_+ and A_- are given by Eqs. (4.6) and (4.7) as

$$\begin{aligned} \Gamma_+(\mu, \nu) = i(2\pi)^{-1}\pi^{-1/2}(\mu - \kappa \sin \theta)(\nu + \nu_0)^{-1/2}(i\beta - \nu_0)^{-1/2}\Gamma(n + 1)A_n(\mu) \\ \cdot \sum_{m=0}^{n} \Gamma^{-1}(m + 1)\Gamma(m + 1/2)\left(\frac{\nu - i\beta}{\nu_0 - i\beta}\right)^m (\beta + i\nu)^{-n-1} \end{aligned} \tag{4.9}$$

$$\begin{aligned} A_-(\mu, \nu) = (2\pi)^{-1}\pi^{-1/2}(\nu - \nu_0)^{1/2}(i\beta - \nu_0)^{-1/2}\Gamma(n + 1)A_n(\mu) \\ \cdot \sum_{m=0}^{n} \Gamma^{-1}(m + 1)\Gamma(m + 1/2)\left(\frac{\nu - i\beta}{\nu_0 - i\beta}\right)^m (\beta + i\nu)^{-n-1} - A_+(\mu, \nu) \end{aligned} \tag{4.10}$$

This completes the solution of the transform equation (3.17), and the remainder of the paper will be devoted primarily to the pressure distribution, as represented by $\Gamma_+(\mu, \nu)$. A similar treatment may be accorded $A_-(\mu, \nu)$ to obtain the downwash off the wing, but it is of less practical importance.

In interpreting the result for $\Gamma_+(\mu, \nu)$, it is convenient to regroup the terms and write, cf. Eq. (2.9b),

$$\Gamma_+(\mu, \nu) = i(\mu - \kappa \sin \theta)\Psi(\mu, \nu) \tag{4.11}$$

$$\begin{aligned} \Psi(\mu, \nu) = (2\pi)^{-1}\pi^{-1/2}(i\nu + i\nu_0)^{-1/2}\Gamma(n + 1)A_n(\mu) \\ \cdot \sum_{m=0}^{n} \Gamma^{-1}(m + 1)\Gamma(m + 1/2)(\beta + i\nu_0)^{-m-1/2}(\beta + i\nu)^{m-n-1} \end{aligned} \tag{4.12}$$

where $\Psi(x, y)$ represents the potential *only* in the sense of Eq. (2.9b).[15]

In principle, the problem is now reduced to quadrature, i.e. the inversion of the transform $\Psi(\mu, \nu)$. Before considering this inversion, it is of interest to infer the behaviour of $\gamma(x, y)$ for small and large y directly from the behavior of $\Gamma_+(\mu, \nu)$ for large and small ν, respectively. Thus, it is found that the pressure behaves as $y^{1/2}$ for small y, since its transform behaves as $\nu^{-3/2}$ for large ν, while it behaves as $y^n \exp(-\beta y)$ for large y, since the transform behaves as $(\beta + i\nu)^{-n-1}$ for small ν, and the initial assumptions are verified. Similarly, it is found that $\alpha_-(x, y)$ behaves as $y^{-1/2}$ for small y and as $((x + y)/-y)^{1/2}\, 1(x, y)$ for large y, therefore vanishing outside the Mach line $x = -y$ and behaving as $(-y)^{-1/2}$ for large (negative) values of y within this Mach line, again verifying the initial assumptions.

[15]Inasmuch as the pressure distribution over the wing is independent of the discontinuity of ψ across the vortex sheet aft of the trailing edge (i.e., the "wake"), it would have been possible to formulate the entire boundary value problem in terms of ψ. This would not be possible for a wing with a subsonic trailing edge.

5. Pressure distribution. It appears that Eq. (4.12) cannot be inverted in finite terms of known functions. However, the result can be partially inverted so as to exhibit characteristic features, after which it will suffice to deal with certain integrals of the pressure.

The ν inversion of Eq. (4.12) may be effected by CF 524.2, CF 526, and the Faltung theorem. The end result is

$$\psi(x, y) = \pi^{-1}1(y) \sum_{m=0}^{n} \binom{n}{m} \Gamma(m + 1/2) \int_0^x d\xi \alpha_n(x - \xi) \cdot \int_0^y d\eta (y - \eta)^{n-m} \eta^{-1/2} k_m(\xi, \eta) \tag{5.1}$$

$$k_m(\xi, \eta) = T_\mu\{(i\nu_0)^{-m-1/2} \exp(-i\nu_0 y)\} \tag{5.2}$$

where $\binom{n}{m}$ is the binomial coefficient, and Eq. (5.2) utilizes the transform convention of footnote 9. Eq. (5.2) may be inverted with the aid of CF 571, CF 866, and the Faltung theorem, yielding

$$k_m(x, y) = \pi^{+1/2}\Gamma^{-1}\left(\frac{1}{2} m - \frac{1}{4}\right) 1(x, x - y) \int_y^x d\xi [(x - \xi)/2\kappa]^{m/2-3/4} \cdot J_{(m/2-3/4)}[\kappa(x - \xi)] J_0[\kappa(\xi^2 - y^2)^{1/2}] \tag{5.3}$$

While Eq. (5.3) appears too complex for direct applications, it establishes the useful fact that $k_m(x, y)$ vanishes for $x < 0$ or $y > x$.

By virtue of the fact that $k_m(\xi, \eta)$ vanishes for $\eta > \xi$, the upper limit y in Eq. (5.1) may be replaced by infinity in the region where $y > x$. Then, if $(y - \eta)^{n-m}$ is expanded by the binomial theorem, and the η integration carried out prior to the μ inversion, the latter may be effected by CF 571 and the end result expressed in terms of powers of y and integrals of Bessel functions of integral and half integral order. This solution is, however, readily obtainable by more direct methods, since the region in question is not influenced by the region between the side edge and the Mach line $y = -x$.

For small values of y the ν inversion of Eq. (4.12) may be brought about by expanding the transform in powers of $(1/i\nu)$, but the results would still involve Bessel functions of fractional (odd multiples of 1/4), therefore being only slightly less complicated than the result of Eq. (5.3).

Still another approach is to expand the solution in powers of the frequency parameter κ. Thus, if we determine the coefficients $c_{pq}^{(m)}$ in the expansion

$$(i\nu_0)^{-m-1/2} \exp(-i\nu_0 y) = (i\mu)^{-m-1/2} \exp(-i\mu y) \sum_{p=0}^{\infty} \kappa^{2p} \sum_{q=0}^{p} c_{pq}^{(m)} (i\mu)^{q-2p} y^q \tag{5.4}$$

the μ inversion of Eq. (5.2) is given by CF 516 as

$$k_m(x, y) = 1(y, x - y) \sum_{p=0}^{\infty} \kappa^{2p} \sum_{q=0}^{p} c_{pq}^{(m)} \Gamma^{-1}\left(2p - q + m + \frac{1}{2}\right) \cdot y^q (x - y)^{2p-q+m-1/2} \tag{5.5}$$

If only the first term ($p = 0$) in this expansion is retained, the summation in Eq. (5.1) is binomial, and the result may be written

$$\psi(x, y) = \pi^{-1}1(y, x - y) \int_0^x d\xi \int_0^y d\eta\, \eta^{-1/2}(\xi - \eta)^{-1/2} \cdot \alpha(x - \xi, y + \xi - 2\eta)1(\xi - \eta) + 0(\kappa^2) \tag{5.6}$$

in agreement with the result obtained in footnote reference 5.

6. Wings of finite aspect ratio. Consider the rectangular planform depicted in Fig. 5,

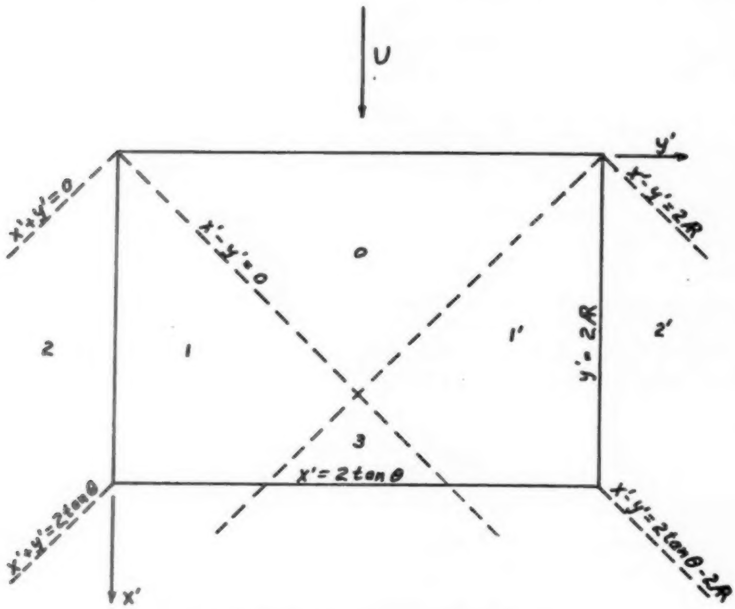

FIG. 5. Rectangular planform and Mach waves.

along with the Mach waves springing from the corners and the regions delineated by these waves. In the dimensionless coordinates of Eq. (2.7), the chord of the wing is chosen as 2 and the span as $2A$. A', the "effective" aspect ratio, is related to the true aspect ration (A) through

$$A' = A \cot \theta \tag{6.1}$$

Due to the choice of the dimensionless coordinates, the Mach waves (or lines) make angles of 45° with the direction of the free stream flow.

A point in region 0 is influenced only by points on the wing, since the fore Mach lines through such a point both intersect the leading edge, so that the downwash is known at every point in the zone of influence of such a point, and the pressure may then be determined by an integration of this downwash. The zone of influence of a point in region I, however, includes part of the region II, off the wing, where the downwash is

not directly prescribed, at least in the problem under consideration. The effect of this latter region is to supply a pressure deficiency in region I, which is just sufficient to cancel the excess pressure along the side edge $y = 0$. Accordingly, if $c_{p_0}(x, y)$ denotes the pressure (jump) coefficient at (x, y) due to the downwash at points on the wing which lie in the fore Mach cone subtended from (x, y), and $c_{p_i}(x, y)$ represents the pressure deficiency due to the points in region II within this Mach cone, the pressure in region 0 is determined by $c_{p_0}(x, y)$ alone, $c_{p_i}(x, y)$ vanishing there, while the pressure in region I is determined by

$$c_{pI}(x, y) = c_{p_0}(x, y) - c_{p_i}(x, y) \tag{6.2}$$

Now, in the case of the quarter infinite wing, the only regions of interest are 0 and I, and the required pressure coefficient indicated by Eq. (6.2) is given by Eq. (5.1), in accordance with the relation, cf. Eq. (2.12),

$$c_p(x, y) = (\rho U^2/2)^{-1}[p(x, y, 0-) - p(x, y, 0+)] \tag{6.3a}$$

$$= 4 \tan \theta \gamma(x, y) \exp [i(\omega t - \kappa x \csc \theta)] \tag{6.3b}$$

That Eq. (6.3) does indeed exhibit a discontinuity (in its derivatives) across the Mach line $x = y$ is indicated by the step function $1(x - y)$, which occurs in the Green's function $k(x, y)$, cf. Eq. (5.3).

The pressure in region I′ on a rectangular wing of finite span may be determined from the pressure in region I by invoking symmetry considerations. Thus, by virtue of the linearization of the problem, the downwash distribution over the wing may be broken down into symmetric and antisymmetric distributions in accordance with the relations

$$\alpha_+^{(s)}(x, y) = \alpha_+^{(s)}(x, 2A - y) \tag{6.4a}$$

$$\alpha_+^{(a)}(x, y) = -\alpha_+^{(a)}(x, 2A - y) \tag{6.4b}$$

and the corresponding pressures determined by $\alpha^{(s)}(x, y)$ and $\alpha^{(a)}(x, y)$ will satisfy similar relations, whence it follows that

$$c_{pI'}(x, y) = c_{p_0}(x, y) \mp c_{p_i}(x, 2A - y) \tag{6.5}$$

the top and bottom signs being associated with the symmetric and antisymmetric problems, respectively. Accordingly, the pressure at any point on a rectangular wing, for which the Mach lines from the leading edge corners do not intersect on the wing (i.e., $A' \geq 2$), or indeed for any wing which has the leading edge $x = 0$, the side edges $y = 0, 2A$, and a trailing edge which lies entirely in the regions 0, I, and I′ and nowhere has a sweepback angle in excess of the Mach angle (45° in the dimensionless coordinates), is directly determined by the solution for the quarter infinite wing.

Consider, now, the less restricted case for which the Mach lines from the leading edge corners do not intersect the opposite side edges (i.e., $A' \geq 1$). For this wing, the pressure coefficient in region III may be cast in the form

$$c_{pIII}(x, y) = c_{p_0}(x, y) - c_{p_i}(x, y) \mp c_{p_i}(x, 2A - y) \tag{6.6}$$

This result follows directly from superposing the pressure deficiencies from the two edges. If the pressure coefficient given by Eq. (6.6) is now integrated over the wing

with a weighting factor which is restricted in its (x, y) dependence to have the same symmetry as $\alpha_+(x, y)$, it is found that the result is identical with that which would have been obtained on the assumption $A' \geq 2$. It follows that the results of this paper may be used for the calculation of the forces and moments on a rectangular wing subject to the restriction $A' \geq 1$.

7. Lift and moments. The integrals of primary interest for the oscillating wing are the lift, mid-chord pitching moment, and rolling moment. The appropriate, dimensionless coefficients are defined as follows:

$$C_L = \left[\frac{1}{2}\rho U^2 S \exp(i\omega t)\right]^{-1} \iint_S [-2p(x, y, 0+)]\, dx\, dy \tag{7.1}$$

$$C_M = \left[\frac{1}{2}\rho U^2 S \cdot 2b \cdot \exp(i\omega t)\right]^{-1} \iint_S (b - x)[-2p(x, y, 0+)]\, dx\, dy \tag{7.2}$$

$$C_l = \left[\frac{1}{2}\rho U^2 S \cdot 2bA \cdot \exp(i\omega t)\right]^{-1} \iint_S (bA - y)[-2p(x, y, 0+)]\, dx\, dy \tag{7.3}$$

where (x, y) are the *true* coordinates, S is the wing area, $2b$ is the chord, and $2bA$ is the span.

Whereas it was more convenient in studying solutions to the potential equation to deal with the quantities defined in Eqs. (2.5)-(2.10), it is rather more natural in dealing with the forces on the rectangular wing to introduce the new, dimensionless quantities

$$x^\square = x/2b = 1/2\, x' \cot\theta \tag{7.4a}$$

$$y^\square = y/b = y' \tag{7.4b}$$

$$\alpha^\square(x^\square, y^\square) \exp(i\omega t) = -U^{-1} \frac{D}{Dt} z(x, y, t) \tag{7.5a}$$

$$\alpha^\square(x^\square, y^\square) = \exp(-i\lambda x^\square)\alpha(x', y') \tag{7.5b}$$

$$\gamma^\square(x^\square, y^\square) \exp(i\omega t) = \left(\frac{1}{2}\rho U^2\right)^{-1}[-2p(x, y, 0+)] = 4bU^{-1}\frac{D}{Dt}\phi(x, y, 0+, t) \tag{7.6a}$$

$$\gamma^\square(x^\square, y^\square) = 4 \tan\theta \exp(-i\lambda x^\square)\gamma(x', y') \tag{7.6b}$$

$$\lambda = 2k \sec^2\theta \tag{7.7}$$

$$k = (\omega b/U) \tag{7.8}$$

The unit of length selected is the half chord (b), in accordance with the generally accepted, two-dimensional notation. The reduced frequency parameters k and λ are also defined in accordance with standard notation, namely that used by Biot.[16] λ should not

[16] loc cit. 1, 2.

be confused with the propagation constant defined by Eq. (3.5) and is henceforth used only in the sense of Eq. (7.7). $\alpha^{\square}$ is the ratio of the (amplitude of the) downwash velocity to the free stream velocity, and $\gamma^{\square}$ is the dimensionless, pressure jump across the wing. In terms of this notation, the substantial time differentiation, cf. Eq. (2.3), is effected by the operator

$$\frac{D}{Dt}(\ ) = (U/2b)\left[\frac{\partial}{\partial x^{\square}}(\ ) + 2ik(\ )\right] \tag{7.9}$$

The dimensionless pressure due the downwash distribution

$$\alpha^{\square}(x^{\square}, y^{\square}) = (y^{\square})^n \alpha_n^{\square}(x^{\square}) \tag{7.10}$$

is given by Eqs. (7.6b), (2.9b), and (5.4e) as

$$\gamma_n^{\square}(x^{\square}, y^{\square}) = 4 \tan \theta 1(y^{\square})\pi^{-1}\left(\frac{\partial}{\partial x^{\square}} + 2ik\right) \sum_{m=0}^{n} \binom{n}{m} \Gamma(m + 1/2)$$
$$\cdot \int_0^{x^{\square}} d\xi \exp(-i\lambda\xi)\alpha_n^{\square}(x^{\square} - \xi) \int_0^{y^{\square}} d\eta\, \eta^{-1/2}(y^{\square} - \eta)^{n-m} k_m(2\xi \tan \theta, \eta) \tag{7.11}$$

The corresponding lift and moment coefficients, as given by Eqs. (7.1)-(7.3) and (7.6) may be written

$$C_{L_n} = \int_0^1 dx\, {}^0\gamma_n^{\square}(x) \tag{7.12}$$

$$C_{M_n} = \int_0^1 dx\left(\frac{1}{2} - x\right){}^0\gamma_n^{\square}(x) \tag{7.13}$$

$$C_{l_n} = \frac{1}{2}\int_0^1 dx\, {}^1\gamma_n^{\square}(x) \tag{7.14}$$

$${}^i\gamma_n^{\square}(x) = A^{-1} \int_0^A dy\, y^i \gamma_n^{\square}(x, y) \tag{7.15}$$

Substituting $\gamma_n^{\square}$ from Eq. (7.11) in Eq. (7.15) and integrating by parts yields

$${}^i\gamma_n^{\square}(x) = 4 \tan \theta\left(\frac{\partial}{\partial x} + 2ik\right) \sum_{m=0}^{n} \binom{n}{m} \Gamma(m + 1/2)$$
$$\cdot \int_0^{x} d\xi \exp(-i\lambda\xi)\alpha_n^{\square}(x - \xi) k_{mn}^i(2\xi \tan \theta) \tag{7.16}$$

$$k_{mn}^0(x) = \pi^{-1}(n - m + 1)^{-1} A^{n-m} \int_0^A dy\, y^{-1/2}(1 - A^{-1}y)^{n-m+1} k_m(x, y) \tag{7.17}$$

$$k_{mn}^1(x) = \pi^{-1}(n - m + 2)^{-1}(n - m + 1)^{-1} A^{n-m} \int_0^A dy\, y^{-1/2}$$
$$\cdot (1 - A^{-1}y)^{n-m+2} k_m(x, y) \tag{7.18}$$

In order to evaluate the $k^i_{mn}(x)$, the restriction $A' \geq 2$ will be imposed (recalling, cf. section 6, that the end results for the force coefficients will be valid for $A' \geq 1$), whence $2x^\square \tan\theta \leq A$. Accordingly, the upper limit A in the y integrations may be replaced by infinity, by virtue of the step function $1(x - y)$ in $k_m(x)$, cf. Eq. (5.3). Then, if $k_m(x, y)$ is substituted in Eqs. (7.17) and (7.18), the y integrals may be evaluated in terms of Gamma functions and the μ inversions effected by CF 571, whence

$$\begin{aligned} k^{0,1}_{mn}(x) &= \pi^{-1/2}\Gamma(n - m + 1) \sum_{s=0}^{n-m+1,2} (-)^s\Gamma^{-1}(n - m - s + 2, 3)\Gamma^{-1}(s + 1) \\ &\cdot \Gamma^{-1}\left[\frac{1}{2}(m + s + 1)\right]\Gamma\left(s + \frac{1}{2}\right)A^{n-m-s}\left(\frac{x}{2\kappa}\right)^{(m+s)/2} J_{(m+s)/2}(\kappa x)1(x) \end{aligned} \tag{7.19}$$

Substituting this result in Eq. (7.16) yields

$$\begin{aligned} {}^i\gamma^\square_n(x) &= 4\tan\theta 1(x)\pi^{-1/2}\Gamma(n + 1)\left(\frac{\partial}{\partial x} + 2ik\right) \sum_{m=0}^{n} \Gamma\left(m + \frac{1}{2}\right)\Gamma^{-1}(m + 1) \\ &\cdot \sum_{s=0}^{n-m+1+i} (-)^s\Gamma^{-1}(n - m - s + 2 + i)\Gamma^{-1}(s + 1)\Gamma^{-1}\left[\frac{1}{2}(m + s + 1)\right] \\ &\cdot \Gamma\left(s + \frac{1}{2}\right)A^{n-m-s} \int_0^x d\xi \exp(-i\lambda\xi)\left(\xi \sin\frac{\theta}{k}\right)^{(m+s)/2} \\ &\cdot J_{(m+s)/2}(\lambda\xi\sin\theta)\alpha_n(x - \xi), \qquad i = 0, 1 \end{aligned} \tag{7.20}$$

8. Downwash independent of y. In principle, any practical downwash distribution may be expanded in terms of the form $y^n\alpha_n(x)$, the only restriction being that the spanwise dependence be sufficiently continuous to allow a power series expansion to the desired accuracy. In practice, the evaluation of the terms in Eq. (7.20) and the integrations indicated by Eqs. (7.12)-(7.14) will be exceedingly cumbersome for large n. The simplest case is that of a downwash which exhibits no spanwise dependence, i.e. $n = 0$, which will be treated in this section. In this case only C_L and C_M are of interest, so that only ${}^0\gamma^\square_0$ is required.

Setting $n = 0$ in Eq. (7.20) and dispensing with the indices yields

$$\begin{aligned} \gamma^\square(x) &= 4\tan\theta 1(x)\left(\frac{\partial}{\partial x} + 2ik\right) \int_0^x d\xi \exp(-i\lambda\xi)\alpha^\square(x - \xi) \\ &[J_0(\lambda\xi\sin\theta) - (\lambda A'\sin\theta)^{-1}\sin(\lambda\xi\sin\theta)] \end{aligned} \tag{8.1}$$

The first term in the square brackets corresponds to the two dimensional result,[17] and the second term may be interpreted as the correction for finite aspect ratio. Moreover, the two dimensional results (for lift and moment coefficients) may be expressed in terms of a set of integrals (required for $n = 0, 1, 2, 3$) of the form[18]

$$f_n = f_n(\lambda, \theta) = \int_0^1 \xi^n \exp(-i\lambda\xi)J_0(\lambda\xi\sin\theta)\, d\xi \tag{8.2}$$

[17]J. W. Miles, *The aerodynamic forces on an oscillating airfoil at supersonic speeds*, J. Aero Sci. **14**, 351-359 (1947), Eqs. (9) and (12).

[18]loc. cit. 2, p. 2.

Accordingly, the two dimensional results are applicable to the rectangular wing (A cot $\theta \geq 1$) if the f_n are replaced by

$$f_n(\lambda, \theta, A') = f_n(\lambda, \theta) - (\lambda A' \sin \theta)^{-1} f_n^*(\lambda, \theta) \tag{8.3}$$

$$f_n^*(\lambda, \theta) = \int_0^1 \xi^n \exp(-i\lambda\xi) \sin(\lambda\xi \sin \theta)\, d\xi \tag{8.4}$$

Integrating Eq. (8.4) twice by parts yields the convenient recursion formula

$$\begin{aligned}\lambda^2 \cos^2 \theta\, f_n^* = \{[(\lambda \sin \theta) \cos(\lambda \sin \theta) + (i\lambda - n) \sin(\lambda \sin \theta)] \exp(-i\lambda) \\ - \delta_n^0(\lambda \sin \theta)\} - 2in\lambda f_{n-1}^* + n(n-1) f_{n-2}^*\end{aligned} \tag{8.5}$$

As a simple example, and as an additional check on the results, the lift coefficient for a flat, rectangular wing at an angle of attack α_0 in a steady flow (so $k = \lambda = 0$) will be calculated. Setting $\alpha^{\square}(x) \equiv \alpha_0$ and $k = \lambda = 0$ in Eq. (8.1) yields

$$\gamma^{\square}(x) = (4\alpha_0 \tan \theta)[1 - (x/A')] \tag{8.6}$$

The corresponding lift coefficient is given by Eq. (7.12b) as

$$C_L = (4\alpha_0 \tan \theta)[1 - (2A')^{-1}] \tag{8.7}$$

in agreement with the well known result of Busemann.[19]

9. Moment due to roll. As an example of an antisymmetric problem, the moment due to roll will be calculated. If the angular velocity in roll about the midspan line is p, the dimensionless downwash distribution is given by

$$\alpha^{\square}(x, y) = \mathbf{1}(x, y)(pbA/U)(A^{-1}y - 1) \tag{9.1}$$

The required pressure function for the determination of the rolling moment, cf. Eq. (7.14b), is therefore $-\,{}^1\gamma_0^{\square} + {}^1\gamma_1^{\square}/A$; calculating this quantity from Eq. (7.23), substituting in Eq. (7.14b), and integrating the resulting terms in $J_1(\lambda\xi \sin \theta)$ and cos $(\lambda\xi \sin \theta)$ by parts, the integrals may be evaluated in terms of the f_n and f_n^* of Eqs. (8.2) and (8.4) with the result

$$\begin{aligned}C_{l_p} &= \partial C_l/\partial(pbA/U) \\ &= -2/3[(\tan \theta + 2ik) f_0 - 2ik f_1] + 2(\lambda A' \sin \theta)^{-1}[(\tan \theta + 2ik) f_0^* \\ &\quad - 2ik f_1^*] - 2(\lambda A' \sin \theta)^{-2}[(\tan \theta + 2ik) f_0 \\ &\quad + 2k \sec \theta (2k \csc \theta - 2i \cos \theta - i \sec \theta) f_1 - 4k^2 \sec \theta \csc \theta f_2 \\ &\quad - \tan \theta \exp(-i\lambda) J_0(\lambda \sin \theta)] - (\lambda A' \sin \theta)^{-3}\{2(\tan \theta + 2ik) f_0^* \\ &\quad + 2k \sec \theta [4k \csc \theta (1 + \cos^2 \theta) - 3i \cos \theta - i \sec \theta] f_1^* \\ &\quad - 4k^2 \sec \theta \csc \theta (1 + \cos^2 \theta) f_2^* - \tan \theta \exp(-i\lambda) \sin(\lambda \sin \theta)\}\end{aligned} \tag{9.2}$$

[19]A. Busemann, *Infinitesimale kegelige Überschallströmung*, Jahrbuch der Luftfahrtforschung **7B**, 105-121 (1943).

10. Arbitrary time dependence. While the present paper has been focused on the case of harmonic time dependence, the results are, in principle, applicable to the calculation of the forces on the wing due to a downwash with an arbitrary time dependence, by virtue of the linearization and the well known Fourier theorem.

An alternative approach to the treatment of arbitrary time dependence would be to use the response to a step function as the basic solution. In this case, the dimensionless downwash is presumed to be of the form

$$\alpha^{\square}(x, y, t) = \alpha^{\square}(x, y)\mathbf{1}[t - t_0(x, y)] \tag{10.1}$$

which is to say that the disturbance at (x, y) arises abruptly at $t_0(x, y)$. Suppose that the solution is placed in the form

$$\gamma^{\square}(x, y, t) = \int_S d\xi \int d\eta\, \mathbf{a}[x - \xi, y - \eta, t - t_0(\xi, \eta)]\alpha^{\square}(\xi, \eta) \tag{10.2}$$

where $\gamma^{\square}$ is the dimensionless pressure jump. Suppose further that the harmonic time dependence problem is written

$$\alpha^{\square}(x, y, t) = \alpha^{\square}(x, y) \exp(i\omega t) \tag{10.3}$$

$$\gamma^{\square}(x, y, t) = \exp(i\omega t) \int_S d\xi \int d\eta\, \mathbf{k}(x - \xi, y - \eta, \omega)\alpha^{\square}(\xi, \eta) \tag{10.4}$$

Then, by virtue of the Fourier representation of the step function of Eq. (10.1), **a** and **k** are related by

$$\mathbf{a}(x, y, t) = (2\pi)^{-1} \lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} d\omega\, (\epsilon + i\omega)^{-1}\mathbf{k}(x, y, \omega) \exp(i\omega t) \tag{10.5}$$

which is to say that the indicial admittance **a** is the Bromwich integral of the Green's function **k**. Accordingly, the two approaches are complementary, and, having solved for **k**, as in the present paper, **a** follows from Eq. (10.5). The application to the gust loading of a rectangular wing has been given in a separate paper.[20]

11. Numerical results. Numerical results for the lifts and moments on a rectangular airfoil due to plunging and pitching oscillations have been obtained and are available elsewhere.[21]

12. Related papers (*added in proof*). The special case of section 7 has been treated independently by Stewart and Li[22] and by Stewartson.[23] The results presented in the latter paper are in agreement with those presented herein, whereas those of the former are not. Moreover, the result (8.1) has been checked by the author, using still a fourth method.[24] Hence, it appears that the general method developed by Stewart and Li may be in error.

[20] J. W. Miles, *Transient loading of supersonic rectangular airfoils*, J. Aero. Sci., **17**, 647-652 (1950).

[21] J. W. Miles and Irven Naiman, *Aerodynamic derivatives for oscillating rectangular airfoils at supersonic speeds*, U.S.N.O.T.S. Tech. Memo RRB-32, Inyokern, Calif. (1949).

[22] H. J. Stewart and T. Y Li, *Periodic motions of a rectangular wing at supersonic speed*, J. Aero. Sci., **17**, 529-538 (1950).

[23] K. Stewartson, *On the linearized potential theory of unsteady supersonic motion*, Q. J. Mech. and Appl. Math. **3**, 182-199 (1950).

[24] J. W. Miles, *On the reduction of unsteady supersonic flow problems to steady flow problems*, J. Aero. Sci. **17**, 64 (1950).

THE AERODYNAMICS OF SUPERSONIC BIPLANES*

BY
W. R. SEARS AND H. S. TAN
Cornell University

1. Introduction. The drag of supersonic wings increases rapidly with increasing thickness. This has led to some speculation about the potentialities of supersonic biplanes, which might afford structural strength and rigidity by virtue of their external structure and hence permit the use of thinner airfoil profiles than would be possible in a monoplane. This brings to mind the possibilities, recognized for several years, of actually reducing the drag of wings by providing the proper wave interactions between the upper and lower wings of a biplane arrangement. That this can be done in the two-dimensional case, i.e., in a biplane of infinite span, was proved in 1935 by Busemann (Ref. 1), who showed that the drag (excluding viscous drag) can be made equal to zero for a biplane at zero lift.

Clearly, it is of interest to study the aerodynamics of finite-span biplanes at supersonic speeds, and especially to estimate the effects of the wing tips on the drag of a finite "Busemann biplane." In this paper we shall report briefly on an investigation (Ref. 8) of the aerodynamics of biplanes having rectangular wings of identical planform. To simplify the work, we shall use here the small-perturbation linear theory, in which all shock and expansion waves are replaced by Mach waves inclined at the free-stream Mach angle. Busemann, to be sure, did not make this approximation in his two-dimensional biplane studies; nevertheless, it should be permissible for the slender airfoils that are of greatest practical interest.

In the linearized theory, the Busemann biplane arrangement becomes the one shown in Fig. 1, i.e., the top and bottom surfaces are flat, the leading-edge Mach wave of either wing intersects the other wing at mid-chord, and the airfoil slopes are related by the formulas, for $x > c/2$,

$$Y_1'(x) = -Y_2'(x - c/2), \qquad Y_2'(x) = -Y_1'(x - c/2).$$

The typical case is then simply that of two isosceles triangles pointing at each other.

In this investigation, the Busemann relationship between gap, chord, and Mach angles shown in Fig. 1 will always be assumed, but it will not be necessary to specify the shape of the profile in deriving some general results. It will be shown that the velocity potential, including all interaction effects, can be calculated by means of integrations involving the wing surface slopes only. The general results will be applied to the numerical calculation of the wave drag, at zero lift, of the typical Busemann arrangement having triangular wing sections.

2. Formulas for source distributions. The equation satisfied by the disturbance velocity potential ϕ in the linearized theory is

$$\beta^2\phi_{xx} - \phi_{yy} - \phi_{zz} = 0; \qquad \beta^2 = M^2 - 1. \tag{1}$$

where subscripts denote partial differentiation with respect to the rectangular Cartesian

*Received March 1, 1950.

coordinates x, y, z. Here M denotes the free-stream Mach number, and the coordinate x is taken in the direction of the undisturbed stream. It has been assumed in deriving Eq. (1) that ϕ_x, ϕ_y, and ϕ_z are small compared to the stream speed, U. A consistent approximate formula for the pressure coefficient is

$$C_p = 2(p - p_0)/\rho_0 U^2 = -2\phi_x/U, \tag{2}$$

where p_0, ρ_0 are the pressure and density of the undisturbed stream.

An elementary solution of Eq. (1) is the so-called supersonic source, $\phi(x, y, z) = [(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2(z - \zeta)^2]^{-1/2}$, provided that the value zero is taken outside

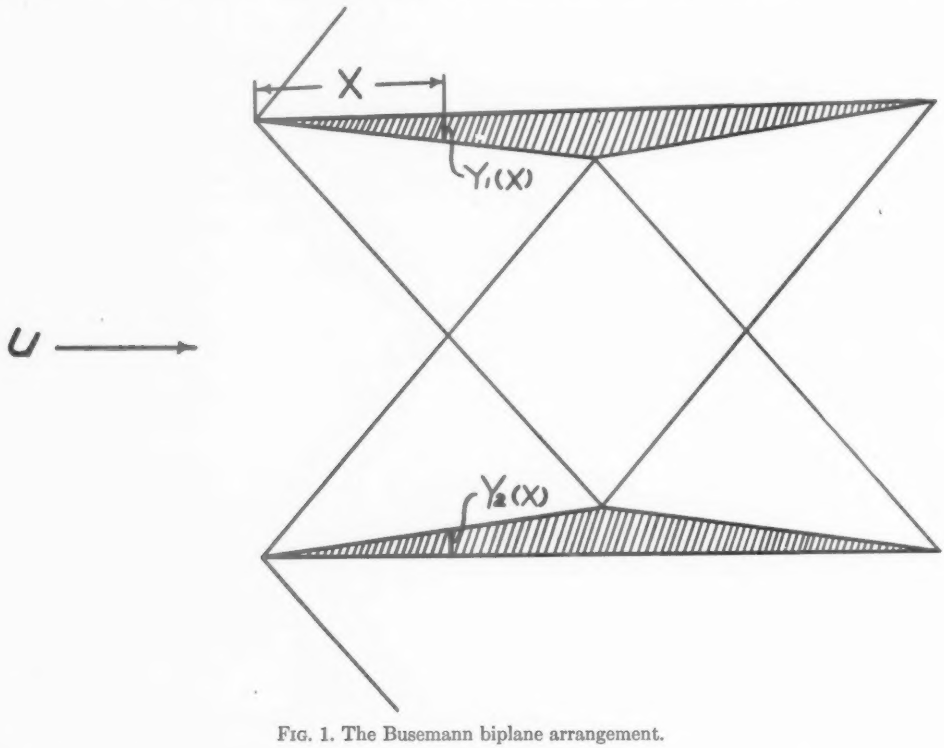

FIG. 1. The Busemann biplane arrangement.

of the Mach cone that originates at the point ξ, η, ζ. For brevity, we shall adopt the following notation:

$$\mu(z) \equiv [(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2]^{-1/2}.$$

It is well known (Refs. 2, 3) that a continuous distribution q of these singularities over a surface parallel to the flow yields a solution satisfying Eq. (1) and the boundary condition $\partial\phi/\partial n = \pi q$ on the surface. Moreover, Evvard (Ref. 4) has shown how a distribution of these sources over a fictitious diaphragm at a wing tip can be used to account for the interaction of upper and lower surfaces of a monoplane wing.

We shall adopt Evvard's scheme here for the calculation of tip effects for both upper

and lower wings, placing a diaphragm at each wing tip and introducing the conditions that these diaphragms are stream surfaces of the flow. The potential at points on the top (T) and bottom (B) surfaces of the upper (u) wing is given by

$$\phi_{uT}(x, y) = -\int_S q_{uT}\mu(0)\, dS, \tag{3}$$

$$\phi_{uB}(x, y) = -\int_S q_{uB}\mu(0)\, dS - \int_{S'} q_{lT}\mu(c)\, dS, \tag{4}$$

and there are analogous formulas for the lower (l) wing. The areas of integration S, on the wing under consideration, and S' on the other wing, are shown in Fig. 2.

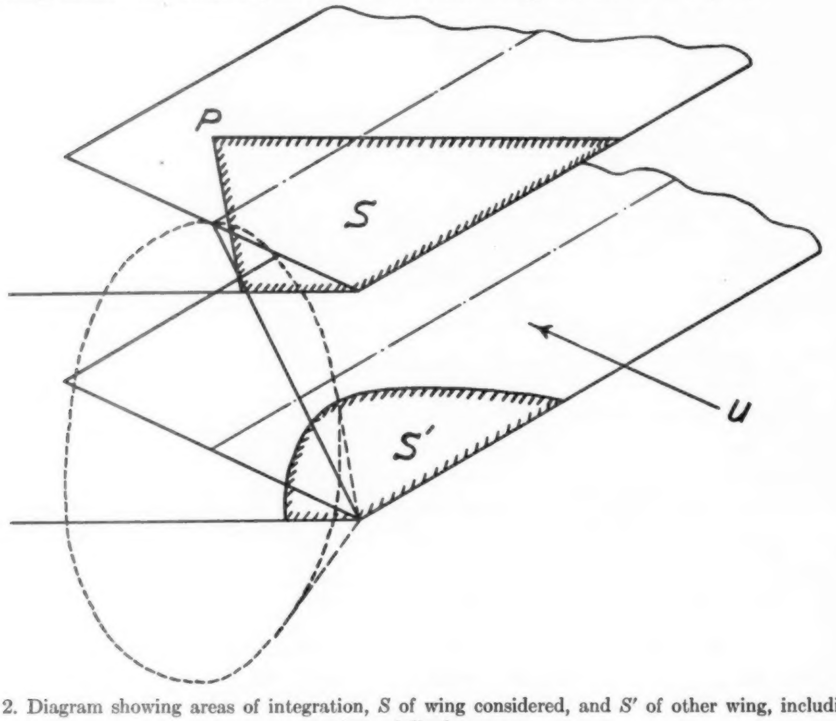

FIG. 2. Diagram showing areas of integration, S of wing considered, and S' of other wing, including portions of diaphragms.

Now the integrations over portions of S and S' can be simplified immediately by use of monoplane results. First of all, it is clear that, in all areas unaffected by biplane interaction, the wing-surface boundary condition requires that $q = U\sigma/\pi$ where σ is the slope of the wing profile in the x direction. Moreover, Evvard has shown, that for monoplanes—and therefore for biplane regions unaffected by interwing interaction—the integration over the diaphragm can be replaced by another integration over part of the wing. For any point forward of mid-chord, i.e., $x < a$, there can be no biplane interaction, hence it is convenient to write the relatively simple expressions for these points before going on to treat the interacting regions.

$x < a$: *no biplane interaction*: Here monoplane results are applicable. For both upper and lower wings, we have (cf. Ref. 4)

$$\phi_T(x, y) = -\frac{U}{\pi} \int_{S_I} \sigma_T \mu(0)\, dS - \frac{U}{2\pi} \int_{S_{I_0}} (\sigma_B - \sigma_T)\mu(0)\, dS, \tag{5}$$

$$\phi_B(x, y) = -\frac{U}{\pi} \int_{S_I} \sigma_B \mu(0)\, dS - \frac{U}{2\pi} \int_{S_{I_0}} (\sigma_T - \sigma_B)\mu(0)\, dS. \tag{6}$$

$x > a$: We consider now a point on the *upper wing, top surface.* If the point lies forward of the Mach line from the tip mid-chord (outside of area N in Fig. 3), there is again

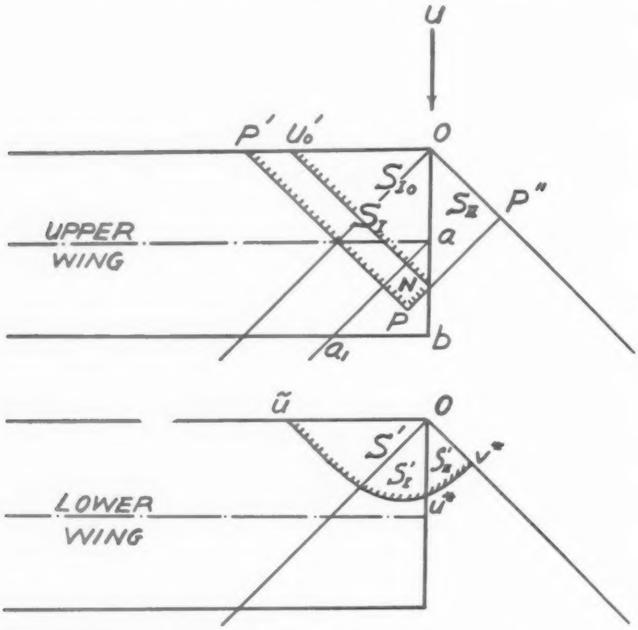

FIG. 3. Diagram defining notation used in calculations for the upper wing.
$aa'b$—N, $PP'Ou_0$—S_I, u_0OP''—S_{II},
$u_0u_0'O$—S_{I_0}, $Ou^*\tilde{u}$—S_I', Ou^*v^*—S_{II}'.

no biplane interference and Eqs. (5) and (6) apply. For a point in N, however, there exists an effect of the lower wing, transmitted through the interaction region of the tip diaphragm. We can write

$$\phi_{uT}(x, y) = -\frac{U}{\pi} \int_{S_I} \sigma_{uT} \mu(0)\, dS - \frac{U}{\pi} \int_{S_{II}} \lambda_u \mu(0)\, dS. \tag{7}$$

Here, and in subsequent formulas, we denote by $\lambda(\xi, \eta)$ the slopes of the tip diaphragms of upper and lower wings. Thus, for any point on the top of the upper-wing diaphragm, q_{uT} is equal to $U\lambda_u/\pi$, and this value has been used in Eq. (7). In regions unaffected by biplane interaction (e.g. for $\xi < a$), $\lambda(\xi, \eta)$ is the same as for a monoplane, and Evvard's

results will be used for such regions. In interacting regions λ is still unknown, of course; its determination constitutes the main problem of this investigation. We shall postpone this to the next section, after writing an analogous formula for points on the bottom surface of the upper wing.

All points of the *bottom surface* of the *upper wing*, for which $x > a$ are affected by biplane interaction. Let S'_I and S'_{II} denote the areas of the lower wing and its diaphragm that affect the point (x, y). The wing-surface boundary condition is

$$q_{uB}(x, y) + \frac{U}{\pi^2} \frac{\partial}{\partial c}\left[\int_{S'_I} \sigma_{lT}\mu(c)\, dS + \int_{S'_{II}} \lambda_l \mu(c)\, dS\right] = \frac{U}{\pi} \sigma_{uB}(x, y). \tag{8}$$

This is an explicit formula for $q_{uB}(x, y)$, involving only known quantities. It may be noted that in the region S'_{II}, q_{lT} has been put equal to $U\lambda_l/\pi$. Moreover, here λ_l is a monoplane value unaffected by biplane interference, and is therefore known from Evvard's work. We now have

$$\begin{aligned} \phi_{uB}(x, y) &= -\int_{S_I + S_{II}} q_{uB}\mu(0)\, dS \\ &\quad - \frac{U}{\pi}\int_{S'_I} \sigma_{lT}\mu(c)\, dS - \frac{U}{\pi}\int_{S'_{II}} \lambda_l \mu(c)\, dS, \end{aligned} \tag{9}$$

where q_{uB} in S_I is known from Eq. (8) and λ_l in S'_{II} is known from monoplane theory. Again the calculation of the diaphragm source distribution, q_{uB} in S_{II}, is postponed to the next section.

For the *lower wing* there are formulas exactly analogous to Eqs. (7), (8) and (9), which will not be written out here.

3. Calculation of diaphragm distributions. The conditions that insure that the tip diaphragms will be stream surfaces are the conditions of equal slope and equal pressure on top and bottom. Since, as Evvard has pointed out (Ref. 5), the diaphragms of a rectangular wing tip are not vortex sheets, equal pressures imply equal values of ϕ, the perturbation velocity potential. We have, then, in region S_{II},

$$\frac{\partial \phi_T}{\partial z} = \frac{\partial \phi_B}{\partial z} \qquad \text{and} \qquad \phi_T = \phi_B\,. \tag{10}$$

The first of these equations leads to

$$\begin{aligned} \frac{U}{\pi}\lambda_u(x, y) &= q_{uT}(x, y) \\ &= -q_{uB}(x, y) - \frac{U}{\pi^2}\frac{\partial}{\partial c}\left[\int_{S'_I} \sigma_{lT}\mu(c)\, dS + \int_{S'_{II}} \lambda_l \mu(c)\, dS\right]. \end{aligned} \tag{11}$$

The second Eq. (10) states that, in S_{II},

$$\begin{aligned} &-\frac{U}{\pi}\int_{S_I} \sigma_{uT}\mu(0)\, dS - \frac{U}{\pi}\int_{S_{II}} \lambda_u \mu(0)\, dS \\ &\quad = -\int_{S_I+S_{II}} q_{uB}\mu(0)\, dS - \frac{U}{\pi}\int_{S'_I} \sigma_{lT}\mu(c)\, dS - \frac{U}{\pi}\int_{S'_{II}} \lambda_l\mu(c)\, dS, \end{aligned} \tag{12}$$

where q_{uB} in S_I and S_{II} is given by Eqs. (8) and (11), respectively. We have now an integral equation for the diaphragm slope λ_u : for points x, y in S_{II} ,

$$2 \int_{S_{II}}^{*} \lambda_u \mu(0)\, dS = \int_{S_I} (\sigma_{uB} - \sigma_{uT})\mu(0)\, dS$$

$$+ \int_{S'_I} \sigma_{lT}\mu(c)\, dS + \int_{S'_{II}} \lambda_l \mu(c)\, dS \tag{13}$$

$$- \frac{1}{\pi} \int_{S_I + S_{II}} \mu(0) \frac{\partial}{\partial c} \left[\int_{S'_I} \sigma_{lT}\mu(c)\, dS^* + \int_{S'_{II}} \lambda_l \mu(c)\, dS^* \right] dS.$$

There is an analogous equation for λ_l , which will not be written out.

Eq. (13) is to be satisfied for all points x, y on the upper-wing diaphragm. For some areas, there is no biplane interaction, i.e., S'_I and S'_{II} vanish, so that the second and third integrals on the right side of Eq. (13) disappear. It is clear that for these points the third integral vanishes as well, since S_I and S_{II} do not contain any points ξ, η affected by interaction. Consequently, for non-interacting points x, y, Eq. (13) reduces to Evvard's integral equation for the diaphragm slope of a monoplane (Ref. 4).

4. Solution of the integral equation. Eq. (13) can be written in the form

$$\int_{S_{II}} \lambda_u \mu(0)\, dS = F_1(x, y) \tag{14}$$

for points x, y in S_{II} , where $2F_1(x, y)$ denotes the entire right-hand side of Eq. (13), and involves only known functions. We now introduce the new coordinates u, v, measured along the two families of Mach lines on the wing in question:

$$u = \frac{M}{2\beta}(\xi + \beta\eta), \qquad v = \frac{M}{2\beta}(\xi - \beta\eta),$$

$$\xi = \frac{\beta}{M}(u + v), \qquad \eta = \frac{1}{M}(u - v),$$

$$J = \frac{\partial(\xi, \eta)}{\partial(u, v)} = -\frac{2\beta}{M^2}, \tag{15}$$

$$\mu(z) \equiv \{(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2\}^{-1/2}$$

$$= \frac{M}{2\beta}\{(u_1 - u)(v_1 - v) - M^2 z^2/4\}^{-1/2}.$$

Our integral equation now takes the form

$$\int_0^{u_1} \frac{du}{(u_1 - u)^{1/2}} \int_u^{v_1} \frac{\lambda(u, v)\, dv}{(v_1 - v)^{1/2}} = F(u_1 , v_1) \tag{16}$$

for points u_1 , v_1 in S_{II} .

The solution can now be found by means of the following process:

$$\int_0^{u'} \frac{F(u_1, v_1)\, du_1}{(u' - u_1)^{1/2}} = \int_0^{u'} \frac{du_1}{(u' - u_1)^{1/2}} \int_0^{u_1} \frac{du}{(u_1 - u)^{1/2}} \int_u^{v_1} \frac{\lambda(u, v)\, dv}{(v_1 - v)^{1/2}}$$

$$= \int_0^{u'} \frac{du_1}{(u' - u_1)^{1/2}} \int_0^{u_1} \frac{H(u, v_1)\, du}{(u_1 - u)^{1/2}}, \tag{17}$$

say,

$$= \int_0^{u'} H(u, v_1)\, du \int_u^{u'} \frac{du_1}{(u' - u_1)^{1/2}(u_1 - u)^{1/2}} = \pi \int_0^{u'} H(u, v_1)\, du.$$

Differentiating this result with respect to u', we have

$$\frac{\partial}{\partial u'} \int_0^{u'} \frac{F(u_1, v_1)\, du_1}{(u' - u_1)^{1/2}} = \pi \int_{u_1}^{v_1} \frac{\lambda(u', v)\, dv}{(v_1 - v)^{1/2}}. \tag{18}$$

We now multiply both sides of Eq. (18) by $(v' - v_1)^{-1/2}$, integrate with respect to v_1, and exchange order of integration in a manner similar to that just employed. The result is

$$\int_{u'}^{v'} \frac{dv_1}{(v' - v_1)^{1/2}} \left(\frac{\partial}{\partial u'} \int_0^{u'} \frac{F(u_1, v_1)\, du_1}{(u' - u_1)^{1/2}} \right) = \pi^2 \int_{u'}^{v'} \lambda(u', v)\, dv \tag{19}$$

which implies (dropping the primes)

$$\lambda(u, v) = \frac{1}{\pi^2} \frac{\partial}{\partial v} \int_u^v \left(\frac{\partial}{\partial u} \int_0^u \frac{F(u_1, v_1)\, du_1}{(u - u_1)^{1/2}} \right) \frac{dv_1}{(v - v_1)^{1/2}}. \tag{20}$$

This solution can be used to calculate the slopes λ_u in regions of interaction. This completes Eq. (7) for ϕ_{uT}, and, by use of Eq. (11), also completes Eq. (9) for ϕ_{uB}. Eq. (20) constitutes a generalization of Evvard's expression for the tip-diaphragm slope, to which, in fact, it immediately reduces when u, v lie in a region free of biplane interaction.

5. Calculation of the potential. Although the biplane problem is now completely solved in principle, the straightforward calculation of ϕ, especially for regions of biplane interference, by substitution in Eqs. (7) and (9), is extremely tedious. Fortunately, as will now be shown, it is possible to eliminate entirely the integration involving λ_u in these two formulas.

In both Eqs. (7) and (9), the term involving λ_u is

$$\int_{S_{II}} \lambda_u \mu(0)\, dS = -\frac{1}{M} \int_0^{v_1} \frac{du}{(u_1 - u)^{1/2}} \int_u^{v_1} \frac{\lambda_u(u, v)\, dv}{(v_1 - v)^{1/2}}, \tag{21}$$

where now u_1, v_1 lie in region S_I.

We return to Eq. (13), which holds for points in S_{II}, and write it in the form

$$\int_0^{u_1} \frac{du}{(u_1 - u)^{1/2}}\, G(u, v_1) = \frac{M\pi}{2U} \phi'(u_1, v_1), \qquad (u_1 \leqq v_1), \tag{22}$$

where

$$G(u, v_1) = \int_u^{v_1} \frac{\lambda_u(u, v)\, dv}{(v_1 - v)^{1/2}} - \frac{1}{2} \int_{-u}^{u} \frac{(\sigma_{uB} - \sigma_{uT})\, dv}{(v_1 - v)^{1/2}}$$

$$- \frac{1}{2U} \int_{-u}^{v_1} \frac{dv}{(v_1 - v)^{1/2}} \frac{\partial}{\partial c} \phi'(u, v) \tag{23}$$

and

$$\phi'(u_1, v_1) = -\frac{U}{\pi}\left\{\int_{S'_I} \sigma_{lT}\mu(c)\, dS + \int_{S'_{II}} \lambda_l \mu(c)\, dS\right\}. \tag{24}$$

Actually, ϕ' is the potential contributed at u_1, v_1 by the lower wing.

The solution of Eq. (22) can be written down immediately (Ref. 6); viz.,

$$G(u, v_1) = \frac{M}{2U}\frac{\partial}{\partial u}\int_0^u \frac{\phi'(u', v_1)\, du'}{(u - u')^{1/2}}, \qquad (u \leqq v_1). \tag{25}$$

Since Eq. (22) is correct only for points u_1, v_1 in S_{II}—i.e. for $u_1 \leqq v_1$—we must restrict u in Eq. (25) as indicated.

Now for points outside of the interaction region, i.e., for $u' \leqq M^2c^2/4v_1$, the interaction potential $\phi'(u', v_1)$ is zero. Thus $G(u, v_1)$ is also zero for $u < M^2c^2/4v_1$.

We can now consider an integral involving $G(u, v_1)$; i.e.,

$$I(u_1, v_1, \kappa) \equiv \int_0^\kappa \frac{du}{(u_1 - u)^{1/2}} G(u, v_1) = \int_{M^2c^2/4v_1}^\kappa \frac{du}{(u_1 - u)^{1/2}} G(u, v_1),$$

where $\kappa \leqq u_1$.

If $\kappa \leqq v_1$ also, $G(u, v_1)$ can be taken from Eq. (25):

$$\begin{aligned} I(u_1, v_1, \kappa) &= \frac{M}{2U}\int_{M^2c^2/4v_1}^\kappa \frac{du}{(u_1 - u)^{1/2}}\left\{\frac{\partial}{\partial u}\int_{M^2c^2/4v_1}^u \frac{\phi'(u', v_1)\, du'}{(u - u')^{1/2}}\right\} \\ &= \frac{M}{2U}\int_{M^2c^2/4v_1}^\kappa \phi'(u', v_1)\left(\frac{\kappa - u'}{u_1 - \kappa}\right)^{1/2}\left[\frac{1}{\kappa - u'} - \frac{1}{u_1 - u'}\right] du' \end{aligned} \tag{26}$$

after some manipulation. Recalling the meaning of $G(u, v_1)$, (Eq. (23)), we can write Eq. (26) as

$$\begin{aligned} &\int_0^\kappa \frac{du}{(u_1 - u)^{1/2}}\int_u^{v_1} \frac{\lambda_u(u, v)\, dv}{(v_1 - v)^{1/2}} \\ &= \frac{M}{2U}\int_{M^2c^2/4v_1}^\kappa \phi'(u', v_1)\left(\frac{\kappa - u'}{u_1 - \kappa}\right)^{1/2}\left[\frac{1}{\kappa - u'} - \frac{1}{u_1 - u'}\right] du' \\ &\quad + \frac{1}{2}\int_0^\kappa \frac{du}{(u_1 - u)^{1/2}}\left\{\int_{-u}^u \frac{(\sigma_{uB} - \sigma_{uT})\, dv}{(v_1 - v)^{1/2}} + \frac{1}{U}\int_{-u}^{v_1} \frac{dv}{(v_1 - v)^{1/2}}\frac{\partial}{\partial c}\phi'(u, v)\right\}. \end{aligned} \tag{27}$$

Since the only restrictions on Eq. (27) are $\kappa \leqq u_1$, and $\kappa \leqq v_1$, it is exactly the result we need for Eq. (21), in which $\kappa = v_1 \leqq u_1$.

We are now prepared to write complete expressions for the potential on top and bottom surfaces of the upper wing, by substitution in Eqs. (7) and (9). Let S_{I_0} be the portion of S_I for which $u \leqq v_1$, as indicated in Fig. 3; then

$$\begin{aligned} \phi_{uT}(x, y) = &-\frac{U}{\pi}\int_{S_I} \sigma_{uT}\mu(0)\, dS - \frac{U}{2\pi}\int_{S_{I_0}} (\sigma_{uB} - \sigma_{uT})\mu(0)\, dS \\ &- \frac{1}{2\pi}\int_{S_{I_0}+S_{II}} \mu(0)\frac{\partial}{\partial c}\phi'(u, v)\, dS \\ &+ \frac{1}{2\pi}\int_{M^2c^2/4v_1}^{v_1} \phi'(u', v_1)\left(\frac{v_1 - u'}{u_1 - v_1}\right)^{1/2}\left[\frac{1}{v_1 - u'} - \frac{1}{u_1 - u'}\right] du', \end{aligned} \tag{28}$$

$$
\begin{aligned}
\phi_{uB}(x, y) = {} & -\frac{U}{\pi} \int_{S_I} \sigma_{uB}\mu(0)\, dS - \frac{1}{\pi} \int_{S_I + S_{II}} \mu(0) \frac{\partial}{\partial c} \phi'(u, v)\, dS \\
& + \frac{U}{2\pi} \int_{S_{I_0}} (\sigma_{uB} - \sigma_{uT})\mu(0)\, dS + \frac{1}{2\pi} \int_{S_{I_0} + S_{II}} \mu(0) \frac{\partial}{\partial c} \phi'(u, v)\, dS \\
& - \frac{1}{2\pi} \int_{M^2c^2/4v_1}^{v_1} \phi'(u', v_1)\left(\frac{v_1 - u'}{u_1 - v_1}\right)^{1/2} \left[\frac{1}{v_1 - u'} - \frac{1}{u_1 - u'}\right] du' \\
& + \phi'(x, y).
\end{aligned} \tag{29}
$$

Formulas (28) and (29) permit the calculation of the potential, and consequently the pressure distribution, on the biplane. It is seen that, whereas we have succeeded in eliminating the integrals involving λ_u, for the upper wing, we are left with integrals involving λ_l, to be taken over certain interaction-free areas. In fact, if interplane interaction of a higher order were encountered, such as an area of the lower wing influenced by interacting regions of the upper wing, it would always be possible to eliminate the λ integral expressing the last stage of tip interaction.

5. Application of results: The wave drag of a finite Busemann biplane at zero lift. The general results obtained here have been applied to one typical practical case, to date.

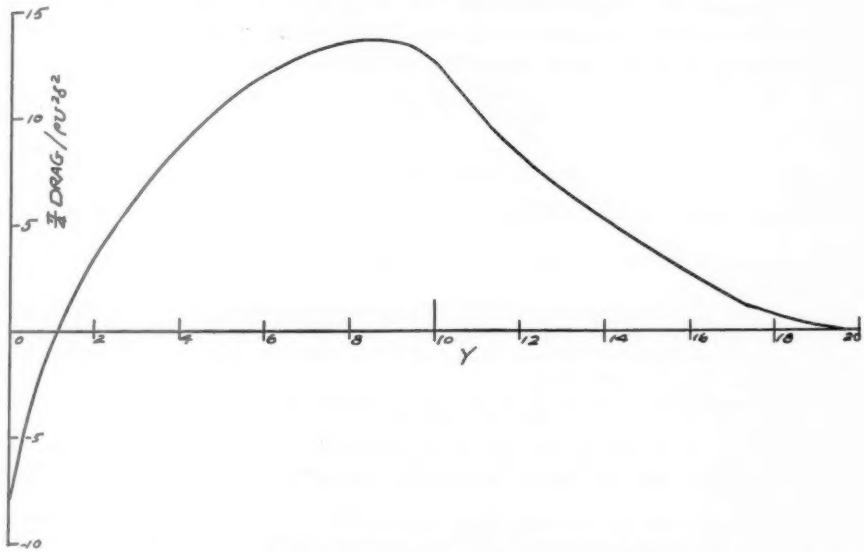

FIG. 4. Spanwise distribution of wave drag near the tip of a rectangular supersonic biplane wing at $M = \sqrt{2}$. The distance y is measured from the tip, and the chord is equal to 20.

This is the case of the finite Busemann biplane, having isosceles-triangular profiles, at zero lift, at a Mach number of $\sqrt{2}$. The computations have been carried out on a computing machine, and integrations have been done by planimeter. These are only preliminary results of an investigation that is still in progress.

These numerical results are shown in Fig. 4, where spanwise distribution of wave drag is plotted. The average drag coefficient of the biplane has been computed from Fig. 4 and found to be

$$C_D = \frac{\text{Total wave drag}}{\rho_0 U^2 S} = 0.823\delta^2/A. \tag{30}$$

where δ denotes the leading-edge angle of the profiles, S the area of one wing, and A the aspect ratio of one wing. It is interesting to compare the wave drag coefficient of a rectangular wing of double-wedge profile, which is

$$C_D = 4\delta^2 \tag{31}$$

For the monoplane, δ denotes the half-angle of the wedge. As would be expected, the ratio of biplane to monoplane wave drags diminishes with increasing aspect ratio.

It is not difficult to show that the force coefficients calculated according to this theory for any Mach number M_1, can be extended to any other value of M by means of the following similarity rule

$$C_D(M) = C_D(M_1)\frac{1 - M_1^2}{1 - M^2} \tag{32}$$

The same correction would apply to the lift coefficient, $C_L(M)$. It is to be understood here that the coefficients $C_D(M)$ and $C_L(M)$ do not apply to the same biplane as $C_D(M_1)$ and $C_L(M_1)$ but to a new configuration proportioned as in Fig. 1 at the Mach number M.

In particular, the result of the present numerical work can be written

$$C_D = 0.823\delta^2/(A\beta^2). \tag{33}$$

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STRUCTURAL ANALYSIS BY DISTRIBUTION OF DEFORMATION*

BY

C. V. KLOUČEK

Prague

The external loading of any structure produces, on the one hand, deformation and on the other hand, stresses which spread from the source of deformation in all directions offered by the given structure; the bending moment may be regarded as a function of stress. The deformation in any arbitrary form, e.g. joint rotation, elongation, etc., represents in reality the single visible and measurable quantity. The expression of the external loading by means of fixed-end moments opened up the way for all the slope deflection solutions, as for instance, equations, balancing methods and the direct distribution of deformation. The balancing methods were based on the idea of alternately locking and unlocking the joints. The validity of both principles presented by Professors G. A. Maney and Hardy Cross follow from the fundamental law of super-position.

In the course of the original derivation of the D. of D.-system (see the author's earlier papers[1] and books[2]), the direct moment distribution methods (Culman, Ritter, Suter etc.) were used as a pattern. Since these older computation systems are no longer currently known, the author here presents the derivation of the two basic D. of D.-relations using newer methods which are first of all briefly characterised.

Slope-deflection equations require first of all the establishment of all fixed-end moments M^{fix} on the assumption that all the joints of the given structure are fixed. After the simultaneous releasing of all joints, the sum of all the adjacent fixed-end moments acts as an impulse on the loaded joint. The deformation waves thus produced circulate through all joints until the complete settling-down of the loaded structure. The state of rest thus attained corresponds to the condition of equilibrium in every joint $\sum M = 0$. If we substitute into this equation the basic relation between the bending moment M_{s-k} and the joint rotations φ,

$$M_{s-k} = \xi_{s-k}(2\varphi_s + \varphi_k) + M_s^{\text{fix}}, \tag{1}$$

then the slope deflection equation for the location-fixed joint $[s]$ and for n-adjacent joints $[k]$ can be written in the form

$$\rho_s\varphi_s + \sum_1^n (\xi_{s-k}\varphi_k) = \sum M_s^{\text{fix}}. \tag{2}$$

The notation introduced here is the same as that used in the author's earlier publications:

$\xi_{s-k} = \xi_{k-s}$—denotes the stiffness factor of the member $[s] - [k]$ and has the value I/L.

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*Received May 26, 1950

[1]Techn. Obzor, Prague, April 1938. Beton und Eisen, Berlin 1939, No. 24. Beton und Stahlbetonbau, Berlin 1943, Nos. 5, 6.

[2]*Rozvod Deformace* I, II, Stat. Naklad. Prague 1940, 1947; *Das Prinzip der fortgeleiteten Verformung* I, W. Ernst & Sohn, Berlin, 1941; *Distribution of deformation. A new method of structural analysis*, C. V. Klouček, Prague, 1949, pp. 510, a limited English edition only for the information of foreign specialists.

ρ_s—denotes the stiffness factor of the joint [s] and equals twice the sum of all member factors, $\rho_s = 2\sum \xi_s$.

θ_s—denotes the actual angular deformation of the joint [s].

$\varphi_s = 2E\theta_s$—denotes only a formal simplification for structures with constant modulus E.

The balancing methods of angle changes by Maney, Goldberg, Cotten, Grinter, Morris, Kammuller, etc. constitute in principle the deformation counterpart to the successive moment distribution method of Hardy Cross. The balancing system e.g. for a structure with three elastic joints may be carried out in various ways e.g. successively [1] − [2], [2] − [3] or throughout the whole structure [1] − [2] − [3], and the like. If the chosen successive system has to be exact, it must express the influence of every joint on every other joint. By means of the sum of the partial angular deformation in any one joint [s], we obtain the resultant rotation φ_s ,

$$\varphi_s = \varphi_s^x + \varphi_s^{xx} + \varphi_s^{xxx} + \text{etc.} \tag{3}$$

The magnitude of the individual increments φ_s^x , φ_s^{xx}, etc. is not a random quantity, but is governed by generally valid laws which can be expressed mathematically.

The direct D. of D.-Method was originally worked out as the deformation counterpart to the direct moment distribution method by E. Suter.[3] The derivation could naturally

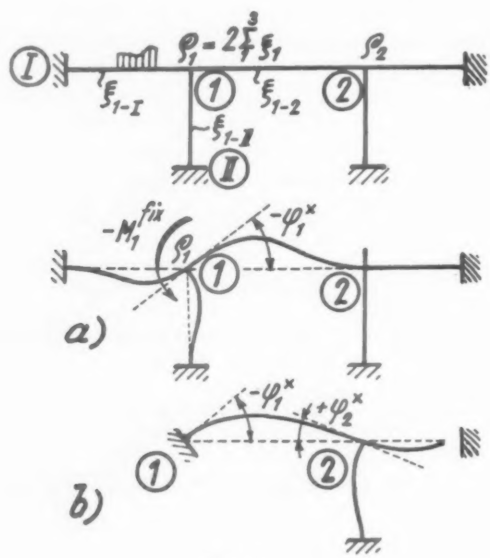

Fig. 1.

be worked out from any of the other exactly valid systems, e.g. according to Clapeyron. Here the two basic relations, for the resultant deformation of the loaded joint and for the direct deformation distribution to the adjacent joint, will be obtained by an analysis of

[3]*Die Festpunktmethode*, 1923.

the balancing principle. The individual deformation increments φ_s^x, φ_s^{xx}, etc., which were originally obtained by successive distribution there and back, will be expressed as resultant sums, thus avoiding the balancing system. The derived relations will be checked by means of methods working with resultant deformations φ_s (e.g. by equations).

Solution for two elastic joints. For the case where a single span of the given structure is loaded (Fig. 1), the initial supposition that every $\varphi = 0$ leads to the determination of the fixed-end moment M_1^{fix}. The first stage of the balancing procedure, i.e. unlocking the loaded joint [1], is governed by the equation for one elastic joint $\rho_1\varphi^x = M_1^{\text{fix}}$; the first partial deformation of the loaded joint [1] is therefore, according to Fig. 1a,

$$\varphi_1^x = \frac{M_1^{\text{fix}}}{\rho_1} . \tag{4}$$

The second stage (Fig. 1b) assumes the locking of the joint [1] in the deformed position for φ_1^x and the unlocking of joint [2]. The relation between the known, non-variable (fixed) value φ_1^x and the unknown φ_2^x is given by the single equation for the second joint $\rho_2\varphi_2^x + \xi_{1-2}\varphi_1^x = 0$. The first partial deformation of the unloaded joint [2] is therefore

$$\varphi_2^x = -\varphi_1^x \frac{\xi_{1-2}}{\rho_2} . \tag{5}$$

After a further alternation of locking we obtain by backward distribution the second deformation increment in the first joint

$$\varphi_1^{xx} = -\varphi_2^x \frac{\xi_{1-2}}{\rho_1} . \tag{6}$$

Successive distribution (there and back) and addition of the corresponding partial values then gives the resultant deformations

$$\varphi_1 = \varphi_1^x + \varphi_1^{xx} + \varphi_1^{xxx} + \text{etc.} \tag{7}$$

$$\varphi_2 = \varphi_2^x + \varphi_2^{xx} + \text{etc.} = -\varphi_1 \frac{\xi_{1-2}}{\rho_2} . \tag{8}$$

From these well-known balancing systems we can easily derive the direct D. of D.-solution for the two elastic joints of Fig. 1. We express the second deformation increment φ_1^{xx} of Eq. (6) in terms of the first deformation φ_1^x of Eq. (5),

$$\varphi_1^{xx} = \varphi_1^x \frac{\xi_{1-2}^2}{\rho_1\ \rho_2} = \alpha_{1-2}\varphi_1^x , \tag{9}$$

where the deformation constant α can arise only between two elastic joints, and its absolute value lies within the limits 0 and 0.25. The next deformation increment φ_1^{xxx} can be expressed by means of Eq. (9) as

$$\varphi_1^{xxx} = \varphi_1^{xx} \frac{\xi_{1-2}^2}{\rho_1\ \rho_2} = \alpha_{1-2}^2\varphi_1^x \tag{10}$$

and the resultant angle φ_1 at the loaded joint is obtained by substituting the partial deformations (9), (10), etc., into Eq. (7),

$$\varphi_1 = \varphi_1^x(1 + \alpha_{1-2} + \alpha_{1-2}^2 + a_{1-2}^3 + \text{etc.}). \tag{11}$$

The deformation process between two elastic joints thus develops—in the same way as for moment distribution[4]—in the form of a simple geometrical series. On expressing the series (for $\alpha_{1-2} < 1$) in the form

$$\left(1 + \sum_{i=1}^{i=\infty} \alpha_{1-2}^{i}\right) = \frac{1}{1 - \alpha_{1-2}} \tag{12}$$

and substituting for φ_1^x from Eq. (4), we obtain the final expression for the loaded joint as in Fig. 1,

$$\varphi_1 = \frac{M_1^{fix}}{\rho_1(1 - \alpha_{1-2})} \, . \tag{13}$$

The direct distribution of this resultant deformation φ_1 to the unloaded joint [2] then has in this case the simplest form according to Eq. (8),

$$\varphi_2 = -\varphi_1 \frac{\xi_{1-2}}{\rho_2} \, . \tag{14}$$

Both the above expressions for φ_1 and φ_2 have been obtained by the addition of all the partial results from the completed balancing process [1] — [2]; the same expressions can be derived by elimination from the two slope-deflection equations. The sign of the individual angular deformations obtained by distribution, direct or successive, follows directly from the visual conception; joints which rotate in a clockwise direction are positive.

Solution for three elastic joints. The determination of the exact effect of a third joint opens up the way for the solution of every straight-beam structure and hence we devote more attention to this problem.

The above balancing system between two adjacent joints can be used successively also for three elastic joints as in Fig. 2. For the temporarily fixed joint [3], the balancing path [1]-[2] gives, according to Eqs. (12), (13) and (14),

$$\varphi_1 = \varphi_1^x\left(1 + \sum_{i=1}^{i=\infty} \alpha_{1-2}^{i}\right) = \frac{M_1^{fix}}{\rho_1(1 - \alpha_{1-2})} \, , \tag{15}$$

$$\varphi_2^x = -\varphi_1 \frac{\xi_{1-2}}{\rho_2} \, . \tag{16}$$

For the fixed joint [1], the successive distribution [2] — [3] of the partial deformation φ_x^2 gives the resultant $\varphi_2 = \varphi_2^x + \varphi_2^{xx}$,

$$\varphi_2 = \varphi_2^x\left(1 + \sum_{i=1}^{i=\infty} \alpha_{2-3}^{i}\right), \tag{17}$$

which after simplification and introduction of φ_1 from Eq. (16) gives

$$\varphi_2 = -\varphi_1 \frac{\xi_{1-2}}{\rho_2(1 - \alpha_{2-3})} \, . \tag{18}$$

The resultant deformation of the third joint is obtained from the known relation

$$\varphi_3 = -\varphi_2 \frac{\xi_{2-3}}{\rho_3} \, . \tag{19}$$

[4]See e.g. Professor H. Cross, Trans. ASCE, **96**, pp. 147, 150, 1932.

When using only the two successive distributions [1] — [2] and [2] — [3], the whole balancing process is not yet at an end; from Fig. 2 it is clear that for instance, the effect

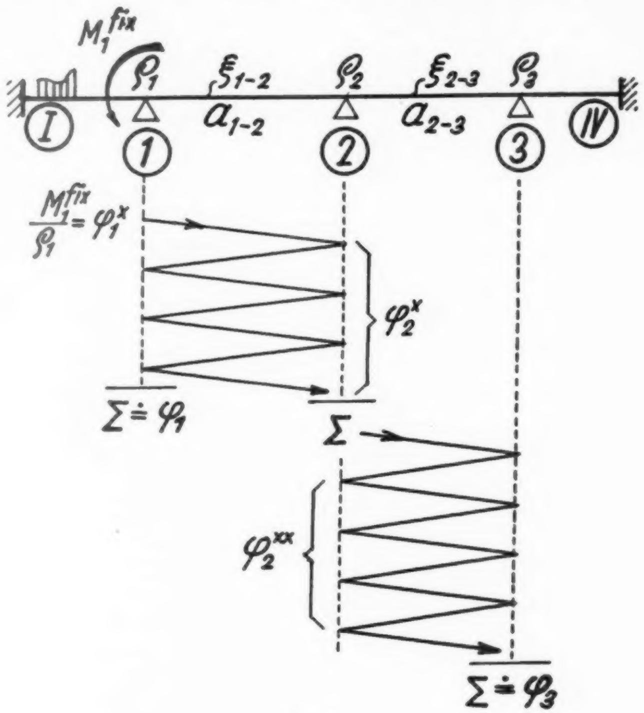

Fig. 2.

of unlocking joint [3] on joint [1] has not been expressed, i.e. that all possible deformation paths according to Fig. 4 have not yet been pursued. Both described successive and direct

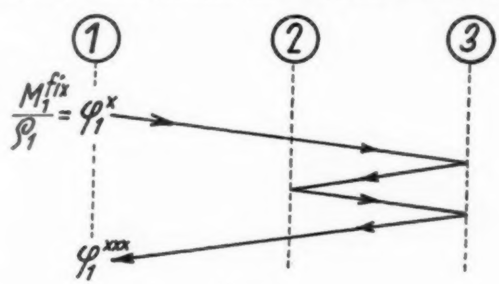

Fig. 3.

solutions cannot exceed a maximum error of 2% for three elastic joints and are therefore well suited for the solution of practical problems; the incomplete balancing process naturally cannot be used for the determination of the exact effect of the third joint.

The creative idea of locking and unlocking the joints can be used in any arbitrary sequence, where the simplest form of the carry-over factor ξ/ρ is successively introduced for the corresponding span. The successive distribution, e.g. in the row 1—2—3—2—3—2—1 as in Fig. 3 can be expressed in terms of the constants α,

$$\varphi_1^{xxx} = \varphi_1^x \frac{\xi_{1-2}\xi_{2-3}\xi_{3-2}\xi_{2-3}\xi_{3-2}\xi_{2-1}}{\rho_2\rho_3\rho_2\rho_3\rho_2\rho_1} = \varphi_1^x\alpha_{1-2}\alpha_{2-3}^2 . \tag{20}$$

In Fig. 4 there are indicated all the distribution paths for three elastic joints which give for example in the loaded joint $\varphi_1 = \varphi_1^x + \varphi_1^{xx} +$ etc. The evaluation of the first

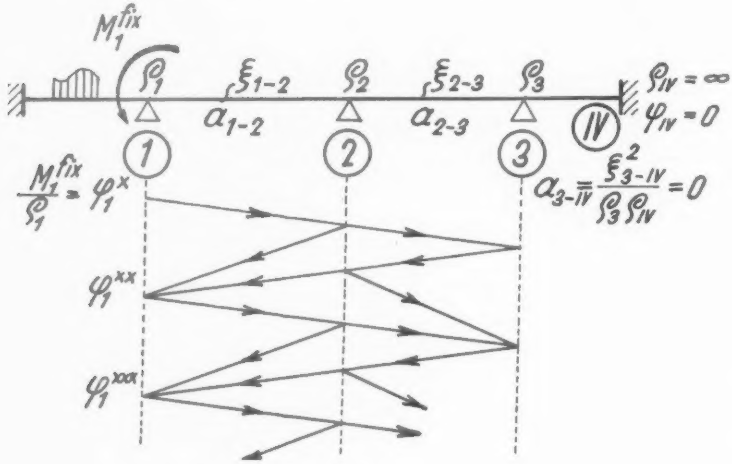

Fig. 4.

deformation increment φ_1^{xx} is possible by only two paths 1—2—1 and 1—2—3—2—1, which, expressed in the sense of Eq. (20), gives

$$\varphi_1^{xx} = \varphi_1^x(\alpha_{1-2} + \alpha_{1-2}\alpha_{2-3}). \tag{21}$$

Then the first rough approximation for $\varphi_1 \doteq \varphi_1^x + \varphi_1^{xx}$ can be written in the form

$$\varphi_1 \doteq \frac{M_1^{fix}}{\rho_1} \{1 + \alpha_{1-2}(1 + \alpha_{2-3})\}, \tag{22}$$

which is in agreement with the exact expression (27) given below on substituting for both indices $i = 1$. In a similar way we could obtain further partial increments in the joint [1] and could derive the final expression for φ_1. This procedure would be exact but considerably time-consuming; the same may be said of the balancing system 1—2—3—2—1—2— etc. according to Table 1.

All the balancing systems for three elastic joints indicated here are only of schematic nature, since they take no account of the various intervals of time required by individual deformation waves. On the assumption that the balancing process is continued until the complete settling-down of the given structure ($i = \infty$), it is naturally immaterial in

what order we add together the individual deformation increments $\varphi = \varphi^x + \varphi^{xx} + \text{etc.}$ While maintaining the basic condition that the effect of every joint on every other joint should be expressed, we can choose the balancing system shown in Fig. 5.

cycle	$-\varphi_1$	carry-over factor	$+\varphi_2$	carry-over factor	$-\varphi_3$
1.	φ_1^x	$\times \frac{\xi_{1-2}}{\rho_2} \longrightarrow$	φ_2^x	$\times \frac{\xi_{2-3}}{\rho_3} \longrightarrow$	φ_3^x
			$\varphi_2^{xx} \longleftarrow$	$\frac{\xi_{3-2}}{\rho_2} \times$	(φ_3^x)
	$\varphi_1^{xx} \longleftarrow$	$\frac{\xi_{2-1}}{\rho_1} \times$	$(\varphi_2^x + \varphi_2^{xx})$		
2.	(φ_1^{xx})	$\times \frac{\xi_{1-2}}{\rho_2} \longrightarrow$	φ_2^{xxx}		
			$(\varphi_2^{xx} + \varphi_2^{xxx})$	$\times \frac{\xi_{2-3}}{\rho_3} \longrightarrow$	φ_3^{xx}
			$\varphi_2^{xxxx} \longleftarrow$	$\frac{\xi_{3-2}}{\rho_2} \times$	(φ_3^{xx})
	$\varphi_1^{xxx} \longleftarrow$	$\frac{\xi_{2-1}}{\rho_1} \times$	$(\varphi_2^{xxx} + \varphi_2^{xxxx})$		
3.	(φ_1^{xxx})	$\times \frac{\xi_{1-2}}{\rho_2} \longrightarrow$	etc.		

Making use of Maney's principle (every $\varphi = 0$) we obtain the fixed-end moment M_1^{fix} and the first value φ_1^x. On using the simple carry-over factor ξ/ρ, the unlocking of joint [2] gives

$$\varphi_2^x = -\varphi_1^x \frac{\xi_{1-2}}{\rho_2} \cdot \tag{23}$$

Completing the balancing of the value φ_2^x between joints [2] − [3] gives according to Eq. (17) the result $\varphi_2^x(1 + \sum \alpha_{2-3}^i)$. After substituting φ_1^x for φ_2^x from the above equation (23), the fixing of joint [2] and unfixing of joint [1] gives the first increment in the loaded joint

$$\begin{aligned} \varphi_1^{xx} &= \varphi_1^x \frac{\xi_{1-2}}{\rho_2} (1 + \textstyle\sum \alpha_{2-3}^i) \frac{\xi_{2-1}}{\rho_1} \\ &= \varphi_1^x \alpha_{1-2}(1 + \textstyle\sum \alpha_{2-3}^i). \end{aligned} \tag{24}$$

In the same way the second and third cycles give

$$\begin{aligned} \varphi_1^{xxx} &= \varphi_1^{xx} \alpha_{1-2}(1 + \textstyle\sum \alpha_{2-3}^i) \\ &= \varphi_1^x [\alpha_{1-2}(1 + \textstyle\sum \alpha_{2-3}^i)]^2 \end{aligned} \tag{25}$$

$$\varphi_1^{xxxx} = \varphi_1^x [\alpha_{1-2}(1 + \textstyle\sum \alpha_{2-3}^i)]^3. \tag{26}$$

The exact value of the resultant deformation follows from the sum $\varphi_1 = \varphi_1^x + \varphi_1^{xx} +$ etc. according to Fig. 5,

$$\varphi_1 = \frac{M_1^{fix}}{\rho_1}\left\{1 + \sum_{i=1}^{i=\infty}\left[\alpha_{1-2}\left(1 + \sum_{i=1}^{i=\infty}\alpha_{2-3}^i\right)\right]^i\right\}. \tag{27}$$

The balancing process for the case of three elastic joints thus obeys the law of a geometri-

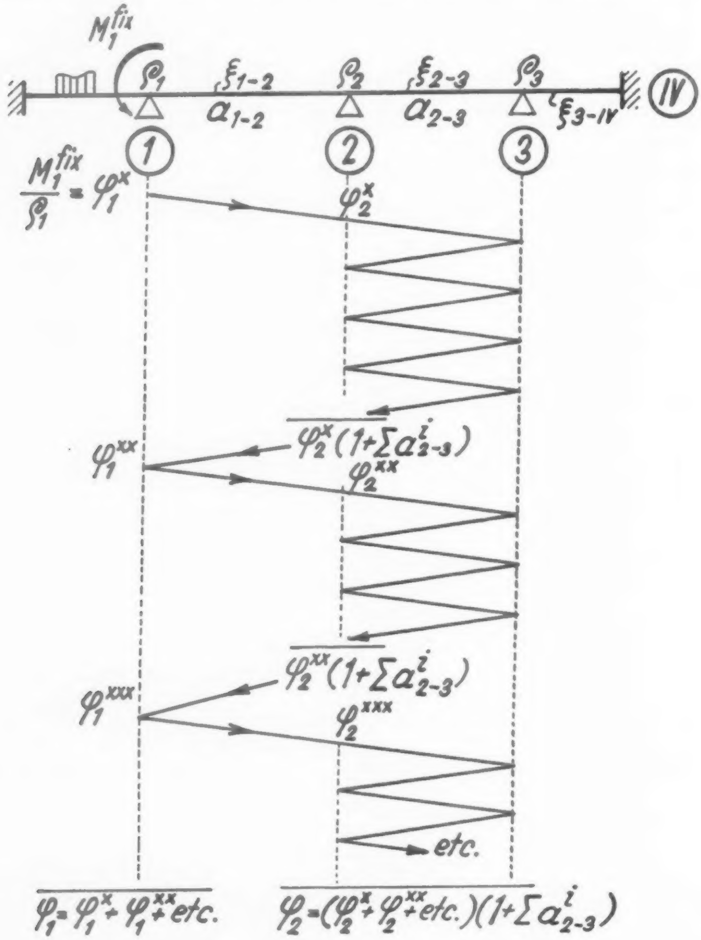

Fig. 5.

cal series of higher order i.e. a series of a series. After expressing the inner series in finite form, the outer series can be expressed in the form

$$1 + \sum_{i=1}^{i=\infty}\left[\frac{\alpha_{1-2}}{1 - \alpha_{2-3}}\right]^i = \frac{1}{1 - \alpha_{1-2}/(1 - \alpha_{2-3})}\ . \tag{28}$$

The exact value for the loaded joint is thus given by the simple expression

$$\varphi_1 = \frac{M_1^{\text{fix}}}{\rho_1[1 - \alpha_{1-2}/(1 - \alpha_{2-3})]} . \tag{29}$$

By addition of the partial values according to Fig. 5, we obtain the exact result for the distributed deformations

$$\varphi_2 = -\varphi_1 \frac{\xi_{1-2}}{\rho_2(1 - \alpha_{2-3})} , \tag{30}$$

$$\varphi_3 = -\varphi_2 \frac{\xi_{2-3}}{\rho_3} , \tag{31}$$

which differ from the previous expressions (18) and (19) only by the exact value φ_1 .

The results (29) to (31) can easily be checked by any other method derived from the final stage of the balancing process, i.e. from the condition of equilibrium $\sum M = 0$. Using the elimination method[5] of Professor J. B. Wilbur, we have for $M_{3-\text{IV}} = -M_{3-2}$, according to Fig. 5,

$$\begin{gathered} \xi_{3-\text{IV}}(2\varphi_3) + \xi_{2-3}(2\varphi_3 + \varphi_2) = 0 \\ \varphi_3 = -\varphi_2 \frac{\xi_{2-3}}{2(\xi_{2-3} + \xi_{3-\text{IV}})} = -\varphi_2 \frac{\xi_{2-3}}{\rho_3} . \end{gathered} \tag{32}$$

The condition of equilibrium $\sum M_2 = 0$ for the joint [2], expressed in terms of deformations

$$\xi_{1-2}(2\varphi_2 + \varphi_1) + \xi_{2-3}\left(2\varphi_2 - \varphi_2 \frac{\xi_{2-3}}{\rho_3}\right) = 0$$

gives, after rearrangement and for $\xi_{2-3}^2/\rho_2\rho_3 = \alpha_{2-3}$,

$$\varphi_2 = -\varphi_1 \frac{\xi_{1-2}}{\rho_2(1 - \alpha_{2-3})} . \tag{33}$$

The condition of equilibrium $M_{1-\text{I}} + M_{1-2} = M^{\text{fix}}$ expressed in terms of deformations

$$\xi_{\text{I}-1}(2\varphi_1) + \xi_{1-2}\left[2\varphi_1 - \varphi_1 \frac{\xi_{1-2}}{\rho_2(1 - \alpha_{2-3})}\right] = M_1^{\text{fix}}$$

gives, after introducing the constant α_{1-2} , the known expression (29),

$$\varphi_1 = \frac{M_1^{\text{fix}}}{\rho_1[1 - \alpha_{1-2}/(1 - \alpha_{2-3})]} . \tag{34}$$

If for the structure of Fig. 5, we introduce an *elastic support* at the *fourth point* (denoted [4] instead of [IV]), the deformation of the loaded joint [1] proceeds according to a geometrical series of the third order. Successive introduction of the finite limits for the individual series results in a continued (chain) fraction, which we denote by the chain value α',

$$\varphi_1 = \frac{M_1^{\text{fix}}}{\rho_1(1 - \alpha'_{1-2})} = \frac{M_1^{\text{fix}}}{\rho_1\{1 - \alpha_{1-2}/[1 - \alpha_{2-3}/(1 - \alpha_{3-4})]\}} . \tag{35}$$

[5] Transactions A.S.C.E., **102**, p. 346 (1937).

In the same way the first deformation distribution to the joint [2] will be

$$\varphi_2 = -\varphi_1 \frac{\xi_{1-2}}{\rho_2(1 - \alpha'_{2-3})} = -\varphi_1 \frac{\xi_{1-2}}{\rho_2[1 - \alpha_{2-3}/(1 - \alpha_{3-4})]} \,. \tag{36}$$

For the case where the *central joint* [2] *is loaded,* we can evaluate the exact value φ_2 by the sum of two balancing systems A and B as shown in Fig. 6. The simple distribution

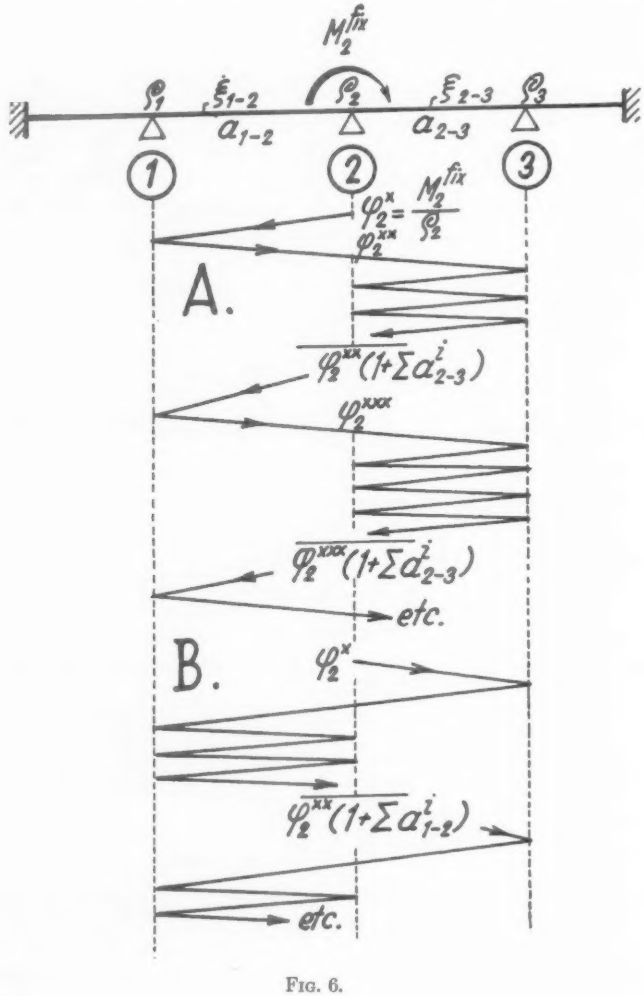

FIG. 6.

[2] — [1] — [2] of the value φ_2^x gives $\varphi_2^{xx} = \varphi_2^x\alpha_{1-2}$ and after completing the balancing [2] — [3] of the value φ_2^{xx}, we have the first increase

$$\varphi_2^{xx}(1 + \sum \alpha_{2-3}^i) = \varphi_2^x\alpha_{1-2}(1 + \sum \alpha_{2-3}^i),$$

and similarly

$$\varphi_2^{xxx}(1 + \sum \alpha_{2-3}^i) = \varphi_2^x[\alpha_{1-2}(1 + \sum \alpha_{2-3}^i)]^2.$$

On interchanging the subscripts, we obtain an analogical expression for the balancing system B. The resultant deformation φ_2 is obtained by the summation of two geometrical series of second order,

$$\varphi_2 = \varphi_2^x\left\{1 + \sum_{i=1}^{i=\infty}\left[\alpha_{1-2}\left(1 + \sum_{i=1}^{i=\infty}\alpha_{2-3}^i\right)\right]^i + \sum_{i=1}^{i=\infty}\left[\alpha_{2-3}\left(1 + \sum_{i=1}^{i=\infty}\alpha_{1-2}^i\right)\right]^i\right\}. \tag{37}$$

When we write all the series in finite form and rearrange the terms,[6] we obtain as the deformation of the central joint [2] the simple expression

$$\varphi_2 = \frac{M_2^{\text{fix}}}{\rho_2(1 - \alpha_{1-2} - \alpha_{2-3})}\,. \tag{38}$$

This relation can be easily checked by means of slope-deflection equations. If we substitute for the two unknowns from the first and third equations

$$\begin{aligned}\varphi_1 &= -\varphi_2\frac{\xi_{1-2}}{\rho_1}\\ \varphi_3 &= -\varphi_2\frac{\xi_{2-3}}{\rho_3}\end{aligned} \tag{39}$$

into the second equation $\rho_2\varphi_2 + \xi_{1-2}\varphi_1 + \xi_{2-3}\varphi_3 = M_2^{\text{fix}}$, the latter may be written in the form

$$\varphi_2 = \frac{M_2^{\text{fix}}}{\rho_2(1 - \xi_{1-2}^2/\rho_1\rho_2 - \xi_{2-3}^2/\rho_2\rho_3)}\,. \tag{40}$$

For *more complex* open *structures* such as that shown in Fig. 7, we obtain by elimina-

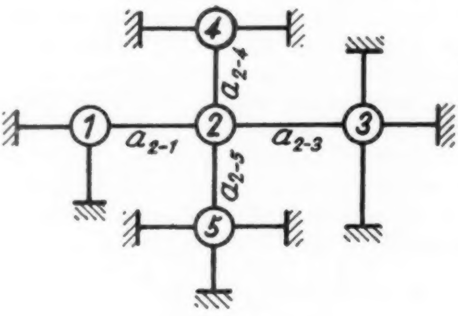

FIG. 7.

tion, in the manner of Eqs. (39) and (40), the expression for the loaded central joint [2],

$$\varphi_2 = \frac{M_2^{\text{fix}}}{\rho_2(1 - \sum_1^4 \alpha_2)}\,, \tag{41}$$

[6]See *Distribution of Deformation*, pp. 26, 31, 32.

which is identical with the approximate formula derived by Mr. C. A. Willson, M.ASCE for a frame network.[7]

For the loading of the outer joint [3] of Fig. 7, we have

$$\varphi_3 = \frac{M_3^{fix}}{\rho_3(1 - \alpha'_{3-2})} \qquad \text{where} \qquad \alpha'_{3-2} = \frac{\alpha_{3-2}}{1 - \alpha_{2-1} - \alpha_{2-4} - \alpha_{2-5}}, \tag{42}$$

and for the fixed joint [3], i.e. for $\alpha_{2-3} = 0$, the distribution of the known (non-variable) deformation φ_3 to the joint [2] gives

$$\varphi_2 = -\varphi_3 \frac{\xi_{3-2}}{\rho_2(1 - \alpha_{2-1} - \alpha_{2-4} - \alpha_{2-5})}. \tag{43}$$

If we summarise now the results of the foregoing analysis, we see that every straight-beam structure can be solved by means of two generally valid relations for the loaded joint [s],

$$\varphi_s = \frac{\sum M_s^{fix}}{\rho_s(1 - \sum_1^n \alpha'_s)} \tag{44}$$

and for the direct (resultant) deformation distribution to the adjacent joint [k].

$$\varphi_k = -\varphi_s \frac{\xi_{s-k}}{\rho_k(1 - \sum_1^{n-1} \alpha'_k)}, \tag{45}$$

where n denotes the number of elastic members meeting in the joint under consideration.

In concluding this paper, the author would like to add the following few remarks:

This article has been devoted only to the theoretical analysis of the basic relations. The solutions for simultaneous loading, closed structures, joint displacement, etc. are given in earlier publications.[2]

The author's attention has been drawn several times to the fact that the solution of linear slope-deflection equations by means of continued fractions is of certain interest from a purely mathematical point of view; the author has not gone into this problem in detail but doubts if the mathematical applicability of the D of D.-system will be so general as, for instance, the Relaxation Method by Professor R. Southwell.

The limited English edition of the work "Distribution of Deformation" has been printed only for specialists abroad; several copies of the book are in various American libraries (e.g. Brown University). The author intends to use the valuable information obtained through contact with specialists in various countries either for the working out of a comparative study or as an addition for the standard edition of the D. of D.-book where also the appropriate literature will be listed.

[7]Transactions A.S.C.E., 102, pp. 352-354 (1937).

—NOTES—

ROTATION OF AN INFINITE PLANE LAMINA: BOUNDARY LAYER GROWTH: MOTION STARTED IMPULSIVELY FROM REST*

BY SWAMI DAYAL NIGAM (*Agra College, Agra, India*)

1. **Introduction.** T. v. Kármán[1] has solved the problem of rotation of an infinite plane lamina in a viscous fluid. He assumes that the motion is steady and the lamina rotates with a constant angular velocity Ω about the axis $r = 0$. He has found exact solutions of the equations of motion which satisfy all the boundary conditions of the problem. The axial velocity does not vanish at infinity, but tends to a finite negative limit, which signifies a steady axial flow towards the rotating lamina. v. Kármán interprets that it is necessary to preserve continuity, since the rotating lamina acts like a centrifugal fan, the fluid moving radially outwards, especially near the lamina.

In the present note I have discussed the growth of motion in the earlier stages of its development caused by an infinite plane lamina which at $t = 0$ is suddenly made to rotate with a constant angular spin Ω about the axis $r = 0$. There grows a boundary layer of thickness proportional to the square root of time, adjacent to the rotating lamina which initially has a zero thickness.

We start with the equations of motion in cylindrical coordinates and substitute in them expressions for u, v, w and p somewhat similar to those used by v. Kármán. Then applying the approximations of the boundary layer theory, we integrate them analytically, satisfying all the boundary conditions required by the problem. The solutions have a serious limitation in that they give initial motion only. They give no information regarding the time after which the steady state is reached.

2. **Equations of motion.** The equations of motion in cylindrical coordinates with terms of azimuthal variation omitted are

$$\nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}$$

$$\nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right] = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} \tag{1}$$

$$\nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}$$

where u, v, w and p are the radial, azimuthal, and axial components of velocity and pressure respectively. The equation of continuity is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 \tag{2}$$

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*Received Oct. 14, 1949.

[1]Goldstein, *Modern developments in fluid dynamics*, vol. I, The Clarendon Press, Oxford, 1938, p. 111.

We now let

$$\begin{aligned} u &= \Omega^2 rtf(\eta) \\ v &= \Omega rg(\eta) \\ w &= -4\nu^{1/2}\Omega^2 t^{3/2} h(\eta) \\ p &= 2\nu\rho\Omega^2 tp(\eta) \end{aligned} \tag{3}$$

where

$$\eta = z/2(\nu t)^{1/2} \tag{4}$$

During early stages of motion when t is small (or in boundary layer theory terminology: when the thickness of the boundary layer is small), we may neglect the terms in the equations of motion containing higher orders of t. Therefore by omitting terms of order t^2 in the equations of motion and continuity, we get to a first order of approximation, the following equations[2]

$$f'' + 2\eta f' - 4f = -4g^2 = -p'' \tag{5}$$

$$g'' + 2\eta g' = 0 \tag{6}$$

$$h'' + 2\eta h' - 6h = -p' \tag{7}$$

$$f = h' \tag{8}$$

3. Solutions of the equations. From (6) we get

$$g = [1 - erf\ \eta] = erfc\ \eta \tag{9}$$

With this solution for g the general solution of Eq. (5) may be expressed[3]

$$f = A(1 + 2\eta^2) + B[(1 + 2\eta^2) erfc\ \eta - 2\pi^{-1/2}\eta e^{-\eta^2}] + 2(\pi^{-1/2}e^{-\eta^2} - \eta\ erfc\ \eta)^2 \tag{10}$$

The boundary condition that $f = 0$ at $\eta = 0$ and $\eta = \infty$ gives

$$A = 0, \qquad B = 2/\pi$$

The function h is obtained by a quadrature of f, and the function p by the double quadrature of $4g^2$. The final analytic expressions for f, h and p, are

$$f = \frac{2}{\pi}[(1 + 2\eta^2) erfc\ \eta - 2\pi^{-1/2}\eta e^{-\eta^2}] - 2(\pi^{-1/2}e^{-\eta^2} - \eta\ erfc\ \eta)^2 \tag{11}$$

$$\begin{aligned} h = \frac{2}{3\pi}[(3\eta + 2\eta^3) erfc\ \eta - 2\pi^{-1/2}(1 + \eta^2)e^{-\eta^2}] - \frac{2\eta}{3}(\pi^{-1/2}e^{-\eta^2} - \eta\ erfc\ \eta)^2 \\ - \frac{2}{3(\pi)^{1/2}} e^{-\eta^2} erfc\ \eta + \frac{2(2)^{1/2}}{3(\pi)^{1/2}} erfc\ 2^{1/2}\eta + \frac{2}{3(\pi)^{1/2}}\left(\frac{2}{\pi} - 2^{1/2} + 1\right). \end{aligned} \tag{12}$$

[2]Goldstein, *loc. cit.* p. 183. A similar approximation has been made there.

[3]The author is indebted to the referee of this paper for the particular solution to Eq. (5).

$$p = (1 + 2\eta^2)(erfc\ \eta)^2 - \frac{4\eta}{\pi^{1/2}} e^{-\eta^2} erfc\ \eta - \frac{2}{\pi} e^{-2\eta^2} + \frac{4(2)^{1/2}}{\pi^{1/2}} \eta\ erfc\ 2^{1/2}\eta\left(-1 + \frac{2}{\pi}\right)$$
$$+ \frac{4\eta}{\pi^{1/2}}\left(\frac{2}{\pi} - 2^{1/2} + 1\right) + \text{const.} \tag{13}$$

Note that there is an anomaly in the behaviour of the pressure which makes it approach infinity as $\eta \to \infty$ and precludes the specification of the constant in the last quadrature by a reasonable physical boundary condition. The anomaly is due to the acceleration of the infinite mass of fluid.

The functions g, f, and h are given in Fig. 1.

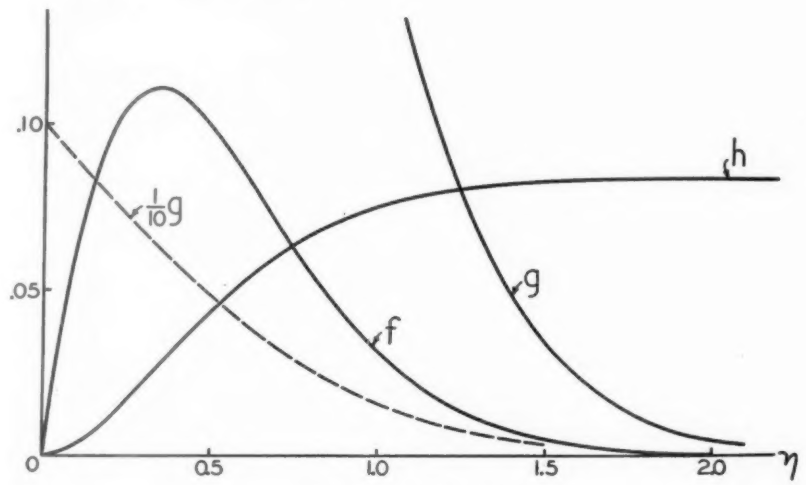

Fig. 1. The flow functions.

4. Stream function. A stream function may be defined by the equations

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z},$$
$$w = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{14}$$

whence the stream function may be expressed

$$\psi = -2\nu^{1/2}\Omega^2 t^{3/2} r^2 h(\eta).$$

This gives stream surfaces which are surfaces of revolution.

In conclusion I express a deep sense of gratitude to Prof. M. Ray, D.Sc. for his kind help in preparation of this note.

RELATION BETWEEN BERGMAN'S AND CHAPLYGIN'S METHODS OF SOLVING THE HODOGRAPH EQUATION*

By T. M. CHERRY (*University of Melbourne*)

When a perfect gas is in steady irrotational isentropic motion in two dimensions, the stream function ψ satisfies a linear differential equation in which the independent variables are components of velocity. For this 'hodograph equation', general forms of solution have been given by Chaplygin[1] and Bergman[2]. The purpose of this note is to show how Bergman's form of solution can be converted into Chaplygin's. Hereby we obtain the specification of the same solution by means of two quite different series, and are in the position to check the extensive computations which are required (in general) to evaluate either of the series.

The results of §1 are due to Chaplygin[1], Lighthill[3] and Cherry[4]; for proofs of the key-formulae (4), (6), (12) reference may be made to [3] or [4]. For Bergman's form of solution the most convenient reference is v. Mises and Schiffer.[5] The different authors use different notations, and the present paper uses a blend of them.

1. Let the rectangular velocity-components be $\tau^{1/2}\cos\theta$, $\tau^{1/2}\sin\theta$, with the unit of speed so chosen that the limiting speed, at which the pressure vanishes, corresponds to $\tau = 1$. Then the hodograph equation is

$$4(1-\tau)\left(\tau^2\frac{\partial^2\psi}{\partial\tau^2} + \tau\frac{\partial\psi}{\partial\tau}\right) + \frac{4\tau^2}{\gamma-1}\frac{\partial\psi}{\partial\tau} + \left(1-\tau-\frac{2\tau}{\gamma-1}\right)\frac{\partial^2\psi}{\partial\theta^2} = 0, \qquad (1)$$

where γ is the adiabatic index of the gas. This equation is soluble by separation of the variables, leading to Chaplygin's form of solution

$$\psi = \sum_\nu c_\nu\psi_\nu(\tau)e^{i\nu\theta}, \qquad (c_\nu \text{ constant}) \qquad (2)$$

where ν can take any real value except $-2, -3, \cdots$, and

$$\psi_\nu(\tau) = \tau^{\nu/2}F(a_\nu, b_\nu; \nu+1; \tau),$$

$$a_\nu + b_\nu = \nu - \frac{1}{\gamma-1}, \qquad a_\nu b_\nu = -\frac{\nu(\nu+1)}{2(\gamma-1)},$$

F denoting the hypergeometric series.

For τ fixed, $\psi_\nu(\tau)$ is a meromorphic function of ν; its poles are at $\nu = -2, -3, \cdots$, and its residue at $\nu = -m$ is $-h_m\psi_m(\tau)$, where

$$h_m = \frac{\Gamma(a_m)\Gamma(1+m-b_m)}{\Gamma(a_m-m)\Gamma(1-b_m)\Gamma(m)\Gamma(1+m)}. \qquad (3)$$

*Received May 15, 1950.

[1] S. A. Chaplygin, Sci. Ann. Univ. Moscow, Phys-Math. Div. Pub. No. 21 (1904).

[2] S. Bergman, N.A.C.A. Tech. Note No. 972 (1945).

[3] M. J. Lighthill, Proc. Roy. Soc. London, (A) 191, 342 and 352 (1947).

[4] T. M. Cherry, Proc. Roy. Soc. London, (A) 192, 45 (1947).

[5] R. v. Mises and M. Schiffer, *Advances in Applied Mechanics*, vol. I, Academic Press, Inc., New York, 1948, p. 249.

For large values of $|\nu|$, $\psi_\nu(\tau)$ changes character at the point $\tau = \tau_s = (\gamma - 1)/(\gamma + 1)$ at which the coefficient of $\partial^2\psi/\partial\theta^2$ in (1) vanishes; it is (for ν real) monotonic for $0 < \tau < \tau_s$, oscillating for $\tau_s < \tau < 1$. In the first case (with which alone we shall be concerned) there is an asymptotic expansion

$$\psi_\nu(\tau) \sim V(\tau)\delta^\nu e^{\nu\lambda}\left\{1 + \sum_1^\infty p_n(\tau)\nu^{-n}\right\}, \tag{4}$$

valid for all complex ν except the negative integers; here

$$\begin{gathered}\lambda = \tau_s^{-1/2} \operatorname{arc\,tanh}\left(\frac{\tau_s - \tau}{1 - \tau}\right)^{1/2} - \operatorname{arc\,tanh}\left(\frac{1 - \tau/\tau_s}{1 - \tau}\right)^{1/2}, \\ V(\tau) = \frac{(1 - \tau)^{1/4+1/2(\gamma-1)}}{(1 - \tau/\tau_s)^{1/4}}, \qquad \delta = \frac{\alpha^\alpha}{(1 + \alpha)^{1+\alpha}}, \qquad 2\alpha = \tau_s^{-1/2} - 1,\end{gathered} \tag{5}$$

and the $p_n(\tau)$ are determinate functions vanishing at $\tau = 0$. Hence follows the partial fraction expansion

$$\psi_\nu(\tau) = \delta^\nu e^{\nu\lambda}\left\{V(\tau) - \sum_{m=2}^\infty \frac{h_m\delta^m e^{m\lambda}\psi_m(\tau)}{m + \nu}\right\}, \tag{6}$$

valid for $0 \leq \tau < \tau_s$ and all ν. If here we formally expand the last factor in powers of ν^{-1} we must obtain (4), and therefore

$$(-1)^n p_n(\tau) V(\tau) = \sum_2^\infty m^{n-1} h_m \delta^m e^{m\lambda}\psi_m(\tau), \qquad (n = 1, 2, \cdots). \tag{7}$$

Substitute (6) in (2) and interchange the order of the double summation. We obtain, for $0 \leq \tau < \tau_s$,

$$\begin{aligned}\psi = V(\tau) \sum_\nu c_\nu \delta^\nu e^{\nu(\lambda+i\theta)} &- \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu}{m + \nu} \\ &- \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu e^{(m+\nu)(\lambda+i\theta)} - c_\nu\delta^\nu}{m + \nu}.\end{aligned}$$

Hence, putting

$$\psi_1(\tau, \theta) = \sum_{m=2}^\infty h_m\delta^m e^{-mi\theta}\psi_m(\tau) \sum_\nu \frac{c_\nu\delta^\nu}{m + \nu}, \tag{8}$$

$$\zeta = \lambda + i\theta, \tag{9}$$

$$\phi_0(\zeta) = \sum_\nu c_\nu\delta^\nu e^{\nu\zeta}, \tag{10}$$

we obtain

$$\begin{aligned}\psi + \psi_1 &= V(\tau) \sum_\nu c_\nu\delta^\nu e^{\nu\zeta} - \sum_{m=2}^\infty h_m\delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta \sum_\nu c_\nu\delta^\nu e^{(m+\nu)t-m\zeta}\, dt \\ &= V(\tau)\phi_0(\zeta) - \sum_2^\infty h_m\delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta e^{m(t-\zeta)}\phi_0(t)\, dt.\end{aligned} \tag{11}$$

For the justification of these manipulations it is sufficient—apart from the over-riding

condition $0 \leq \tau < \tau_s$—that (i) for all values of ν comprised in (2) and all positive integers m, $| \nu + m |$ has a positive lower bound, and (ii) the series $\sum | c_\nu \delta^\nu e^{\nu \zeta} |$ converges in some strip $\zeta_1 \leq \text{Re } \zeta \leq 0$; the proof rests essentially upon the estimations

$$\psi_\nu(\tau) = V(\tau)\delta^\nu e^{\nu\lambda}\{1 + O(\nu^{-1})\}, \qquad 2\pi h_m = \delta^{-2m}\{1 + O(m^{-1})\}, \tag{12}$$

of which the former is the first approximation derived from (4).

We note that $\psi_1(\tau, \theta)$, as defined in (8), is a solution of (1) in Chaplygin's form.

2. To convert (11) into Bergman's form of solution we expand the factor $e^{m(t-\zeta)}$ and rearrange the resulting double sum. After an appeal to (7) this gives

$$\psi + \psi_1 = V(\tau)\phi_0(\zeta) - \sum_2^\infty h_m \delta^m e^{m\lambda}\psi_m(\tau) \int_0^\zeta \phi_0(t)\, dt \sum_1^\infty m^{n-1}(t - \zeta)^{n-1}/(n-1)!$$

$$= V(\tau)\phi_0(\zeta) - \int_0^\zeta \sum_{n=1}^\infty (t - \zeta)^{n-1}\phi_0(t)\, dt/(n-1)! \cdot \sum_{m=2}^\infty m^{n-1} h_m \delta^m e^{m\lambda}\psi_m(\tau)$$

$$= V(\tau)\phi_0(\zeta) + \int_0^\zeta \sum_{n=1}^\infty \frac{(\zeta - t)^{n-1} p_n(\tau) V(\tau)}{(n-1)!}\, \phi_0(t)\, dt. \tag{13}$$

The transformation is valid provided the series

$$\sum_2^\infty h_m \delta^m e^{m\lambda} \psi_m(\tau) e^{m|\zeta|}$$

converges absolutely, and by (12) this is secured if $| \zeta | + 2\lambda$ is negative; hence from (9), it is sufficient that λ be negative (as it is for $0 < \tau < \tau_s$) and that

$$-3^{1/2} | \lambda | < \theta < 3^{1/2} | \lambda |. \tag{14}$$

On the left of (13) $\psi + \psi_1$ is a solution of Chaplygin's form, and on the right we have this expressed in Bergman's form* in terms of an arbitrary analytic function $\phi_0(\zeta)$. The identification not merely of form but of content will be complete provided Bergman's G_n and the present p_n are related by

$$G_n = (-2)^n\, p_n. \tag{15}$$

Now if, as in [5], we examine the conditions that the form on the right of (13) be a solution, with $\phi_0(\zeta)$ remaining arbitrary, we find that the derivative of p_n *must* be determined entirely by p_{n-1}, so that p_n is determined apart from an additive constant. This constant is, in the preceding work, determined by the condition $p_n = 0$ for $\tau = 0$, while in [5] the condition is taken to be $G_n = 0$ for $\lambda = -\infty$; and these conditions agree since to $\tau = 0$ corresponds by (5) $\lambda = -\infty$. Hence (15) expresses merely the same function in two different notations.

In conclusion, it may be remarked that the conditions assumed in proving (13) are, in one respect, more restrictive than those which validate Bergman's form of solution on the right; for our conditions imply that $\phi_0(\zeta)$ is regular in a strip $\zeta_1 < \text{Re } \zeta < 0$, whereas Bergman requires only regularity in a partial neighbourhood of $\zeta = 0$. Against this must be set the fact that Bergman's form is established only when θ is restricted as in (14), while in the Chaplygin form θ is unrestricted.

*See particularly [5], p. 258, un-numbered equation following (4); here ψ^* is defined in (1.6), where $z^{-1/2}$ is the same as $V(\tau)$ of the present paper.

THE DESIGN OF TWO-DIMENSIONAL CONTRACTION SECTIONS*

By PAUL A. LIBBY and HOWARD R. REISS (*Polytechnic Institute of Brooklyn*)

1. Introduction. The design of the expansion section of two-dimensional or axially symmetric supersonic nozzles has been the subject of considerable investigation. The methods to be used in their design are widely understood and readily applied. These are all based on the assumption of uniform, locally sonic flow at the minimum section. However, the literature on the design of a contraction section which will produce this uniform flow appears to be limited.

Tsien [1], Szczeniowski [2] and Smith and Wang [3] have presented methods for the design of axially symmetric contraction cones for an inviscid, incompressible fluid. There are, however, cases in which the contraction section is essentially two-dimensional and in which the inlet flow is approximately axial and uniform. For such contraction sections the authors were able to find no literature, although the above referenced methods could be applied to this case. However, the hodograph method [4] for incompressible, two-dimensional potential flow appears to offer a simpler solution to this problem.

2. Development. In this method the actual contraction section is taken from an infinite channel with asymptotes of $y = \pm\alpha$ at $x \to -\infty$ and of $y = \pm\beta$ at $x \to \infty$. The velocity at these extremes is in the x direction and is equal to a and b respectively. It is required to find a contour connecting these asymptotes in such a fashion that the resultant velocity is everywhere monotonically increasing in the direction of flow. This requirement assures for both the incompressible and compressible flow that boundary layer separation and local compressibility effects will be avoided (cf. reference 1).

Although there may be chosen an infinite variety of functions, which give streamlines connecting points $(a, 0)$ and $(b, 0)$ in the hodograph (u, v) plane and which satisfy this requirement, a simple set may be found as follows: Consider the complex potential $F = \phi + i\psi$

$$F(\overline{w}) = c[\ln(\overline{w} - b) - \ln(\overline{w} - a)], \tag{1}$$

where $\overline{w}$ is the usual complex conjugate velocity, $u - iv$, and where c is a positive, real constant. Considering the real and imaginary parts of Eq. (1), one finds that

$$\phi/c = (1/2)\ln\{[(u-a)^2 + v^2]/[(u-b)^2 + v^2]\}, \tag{2a}$$

$$\psi/c = \tan^{-1}[-v/(u-a)] - \tan^{-1}[-v/(u-b)]. \tag{2b}$$

It is of interest to examine the streamlines in the $\overline{w}$ plane. From Eq. (2b) one obtains the equation

$$\psi/c = \theta_a - \theta_b = \gamma,$$

where these angles are shown in Fig. 1. An examination of this figure indicates that the streamlines are arcs of circles passing through the points $(a, 0)$ and $(b, 0)$. Since the velocity at any point on such a streamline is represented in the $\overline{w}$ plane by a vector to that point from the origin, only those streamlines corresponding to $\pi/2 \leq |\psi/c| \leq \pi$ will yield wall shapes in the z plane with monotonically increasing velocities. The arcs

*Received June 5, 1950.

corresponding to $|\psi/c| < \pi/2$ yield, in the z plane, streamlines along which the velocity is locally decreasing in two regions. Figure 1 also indicates that the centerline of sym-

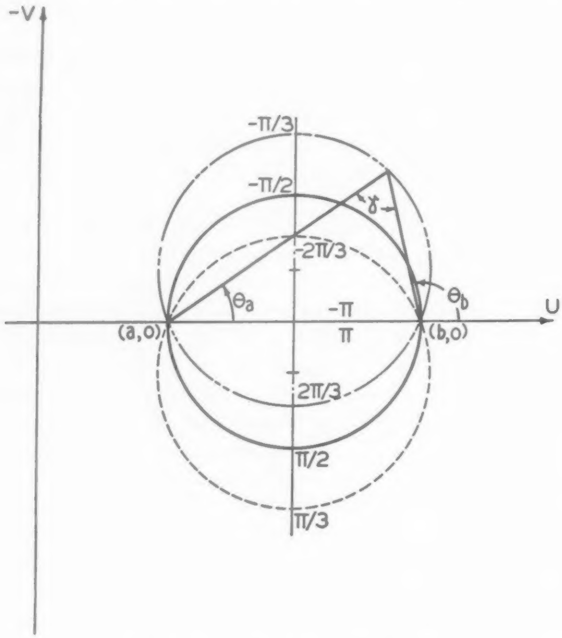

FIG. 1. Streamlines in the $\overline{w}$ plane.

metry of the contraction section corresponds to $\psi/c = \pm\pi$. The x and y coordinates of the streamlines will now be determined.

Solving for $\overline{w}$ from Eq. (1), one obtains

$$\overline{w} = [b - a \exp(-F/c)]/[1 - \exp(-F/c)]. \tag{3}$$

From the definition of F, the complex variable $z = x + iy$ is given by the transformation equation

$$z = \int (\overline{w})^{-1} \, dF + C_0 , \tag{4}$$

or from Eq. (3)

$$z = \int \{[1 - \exp(-F/c)]/[b - a \exp(-F/c)]\} \, dF + C_0 , \tag{5}$$

which, upon integration gives

$$z = (F/b) - (c/b)[(b/a) - 1] \ln [(b/a) - \exp(-F/c)] + C_0' , \tag{6}$$

where C_0 and C_0' are arbitrary complex constants of integration.

Now with $F = \phi + i\psi$ one obtains, by equating the real and imaginary parts of each side of Eq. (6),

$$x = (\phi/b) - (c/b)[(b/a) - 1] \ln \{1 + [a \exp(-\phi/c)/b]^2 - [2 a \cos(\psi/c) \exp(-\phi/c)/b]\}^{1/2} + C_1, \tag{7a}$$

$$y = (\psi/b) - (c/b)[(b/a) - 1] \cdot \tan^{-1} \{[\exp(-\phi/c) \sin(\psi/c)]/[(b/a) - \exp(-\phi/c) \cos \psi/c]\} + C_2, \tag{7b}$$

where again C_1 and C_2 are arbitrary real constants. To obtain the streamlines of the flow given by Eqs. (7a) and (7b) one may consider them, for ψ = constant, two parametric equations giving x and y in terms of ϕ.

To determine c and C_2, Eqs. (7a) and (7b) are examined. For $\phi \to -\infty$, $x \to -\infty$ provided $b/a > 1$, and

$$y \to (\psi/b) - (c/b)[(b/a) - 1][\pm\pi - (\psi/c)] + C_2, \tag{8a}$$

where the upper and lower signs correspond to $\psi/c \gtrless 0$ respectively. Furthermore, for $\phi \to \infty$, $x \to \infty$,

$$y \to (\psi/b) + C_2, \tag{8b}$$

where the arctangent is chosen in the first and second quadrant for $\psi/c > 0$ and in the third and fourth quadrant for $\psi/c < 0$.

The constant C_2 is determined so that $\psi/c = \pm\pi$ along the x axis ($y = 0$). Thus from Eq. (7b)

$$C_2 = \mp c\pi/b, \tag{9a}$$

giving as $\phi \to -\infty, \quad y \to [(\psi/c) \mp \pi]/(a/c),$ (9b)

and as $\phi \to \infty, \quad y \to [(\psi/c) \mp \pi]/(b/c).$ (9c)

Now if it is required that as $x \to -\infty$ $(\phi \to -\infty)$, $y \to \mp\alpha$ for $\psi/c = \pm\psi_0$, i.e., that the contraction section walls correspond to $\psi/c = \pm\psi_0$, then from Eq. (9b)

$$\alpha = (-\psi_0 + \pi)/(a/c)$$

or

$$c = a\alpha/(-\psi_0 + \pi). \tag{10a}$$

Then as $x \to \infty$ $(\phi \to \infty)$, from Eqs. (9c) and (10a) it follows that

$$y \to \mp a\alpha/b \equiv \mp\beta. \tag{10b}$$

Thus if $\Phi \equiv (a/b) \exp(-\phi/c)$, Eqs. (7a) and (7b) with Eqs. (9a), (10a) and (10b) yield for the wall contour in the upper half of the z plane ($\psi/c = -\psi_0$) denoted by the subscript zero

$$\xi_0 = (x_0/\alpha) = -[(\beta/\alpha)/(-\psi_0 + \pi)]\{\ln \Phi + 1/2[(\alpha/\beta) - 1] \ln(1 + \Phi^2 - 2\Phi \cos \psi_0)\} \tag{11a}$$

and

$$\eta_0 = (y_0/\alpha) = [(\beta/\alpha)/(-\psi_0 + \pi)]$$
$$\cdot \{-\psi_0 - [(\alpha/\beta) - 1] \tan^{-1} [-\Phi \sin \psi_0/(1 - \Phi \cos \psi_0)] + \pi\}. \tag{11b}$$

The arbitrary constant C_1 in Eq. (7a) leads only to a translation of the streamlines along the x or ξ axis and thus may conveniently be set equal to zero.

Equations (11a) and (11b) give the final contour in the upper half plane in terms of the parameter $\Phi(0 \leq \Phi < \infty)$. Note that $\xi_0 \to -\infty$ and $\eta_0 \to 1$ as $\Phi \to \infty$, and $\xi_0 \to \infty$ and $\eta_0 \to \beta/\alpha$ as $\Phi \to 0$. In order to obtain the streamlines inside the wall contour, ψ/c may be permitted to vary from its wall value, $-\psi_0$, to $-\pi$. It might be pointed out that for asymmetrical channels ψ_0 would simply take on different values in the upper and lower halves of the z plane.

3. Numerical Example. In Fig. 2 the wall contours for $\alpha/\beta = 4$ with $\psi_0 = \pi/2$, $2\pi/3$ and $\pi/3$, are shown. Note that the latter contour, for which the velocity is not

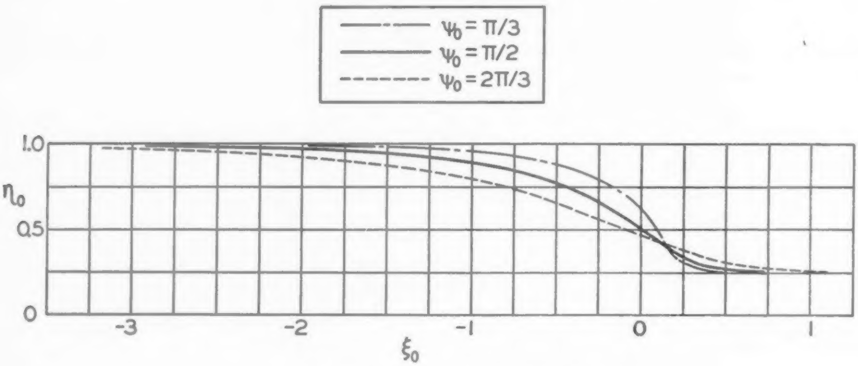

FIG. 2. Wall contours. Contraction ratio = 4.

monotonically increasing, appears to be reasonable and might well be drawn and selected for a contraction cone if no analysis were carried out. It will be noted that the asymptotic values, 1 and β/α are approached quickly. Criteria of closeness to the asymptotes, of uniformity of the u velocity, or of the smallness of the v velocity can be applied to establish the values of ξ_0 determining the length of the section.

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A NOTE ON BATEMAN'S VARIATIONAL PRINCIPLE FOR COMPRESSIBLE FLUID FLOW*

By CHI-TEH WANG (*New York University*)

1. Introduction. The variational principle for a compressible fluid was first studied by Hargreaves [1], who showed that the integrand of the variational integral is a linear function of the pressure. A variational principle for an inviscid compressible fluid was formulated by Bateman [2]. A study of Bateman's work, however, shows that his variational principle is applicable only when the domain of the flow is finite. A large class of aerodynamic problems require the study of a flow field which extends to infinity. In such cases, Bateman's principle must be modified. This fact has already been noted in references [3] and [4], in which the Rayleigh-Ritz method was used in the approximate solution of compressible flows past arbitrary bodies. The formulation of a suitable variational principle in these references was however carried out in connection with the particular problems considered, so that the derivation appears to be in a rather restricted form. In this note, a more general formulation is presented and the resulting variational integral is written in a more general form. The author is indebted to Professor K. O. Friedrichs for his kind suggestions and discussions.

2. Bateman's variational principle. For steady, inviscid, irrotational compressible flow, the governing differential equation is

$$\left[a^2 - \left(\frac{\partial \phi}{\partial x_i}\right)^2\right] \frac{\partial^2 \phi}{\partial x_i\, \partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i\, \partial x_j} = 0, \tag{1}$$

where

$$a^2 = \frac{\gamma - 1}{2}\left(q_m^2 - \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_k}\right). \qquad (i, j, k = 1, 2, 3). \tag{2}$$

In Eqs. (1) and (2), a is the velocity of sound, ϕ is the velocity potential, x_i are the Cartesian coordinates, γ is the ratio of specific heats, q_m is the maximum attainable velocity in the flow. A repetition of the subscripts in the above expression indicates summation.

Bateman's problem is to show that the variational integral

$$I_1 = \int_V p(\phi)\, dV \tag{3}$$

has Eq. (1) as its Euler's equation, where p is the pressure, $d\,V$ is the elementary volume, and the integration is extended to the whole volume of the fluid.

For barometric fluid, p may be written as

$$p = A + B\rho^\gamma,$$

where ρ is the density, and A, B are constants. In terms of ϕ, one obtains

$$p = A + C\left(q_m^2 - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i}\right)^k \tag{4}$$

*Received June 5, 1950.

where

$$C = B^{-1/(\gamma-1)}(\gamma - 1/2\gamma)^k \qquad \text{and} \qquad k = \gamma/(\gamma - 1).$$

With the expression of p as given by (4), the first variation of I_1 may be carried out with respect to ϕ and the condition $\delta I_1 = 0$ leads to

$$\int_V \delta\phi \frac{\partial}{\partial x_i}\left(\rho \frac{\partial \phi}{\partial x_i}\right) dV + \int_{S_1+S_2} \delta\phi\, \rho \frac{\partial \phi}{\partial n}\, dS = 0, \tag{5}$$

where δ indicates first variation, $\partial/\partial n$ is the derivative in the inward normal direction, $S_1 + S_2$ is the surface that encloses the volume V, S_1 denotes the stream surface and S_2 denotes the boundary surface at infinity. Since $\delta\phi$ is arbitrary in V, the condition $\delta I_1 = 0$ gives the continuity equation

$$\frac{\partial}{\partial x_i}\left(\rho \frac{\partial \phi}{\partial x_i}\right) = 0$$

as the Euler's equation of the variational integral (3).

If the domain is finite and the boundary surfaces are stream surfaces, no condition has to be imposed on ϕ and $\partial\phi/\partial n = 0$ follows as the natural boundary condition. If the domain is infinite, $\partial\phi/\partial n = 0$ on S_1 can still be concluded from the condition $\int_{S_1} \delta\phi\, \rho(\partial\phi/\partial n)\, dS = 0$. At infinity ϕ must be prescribed, and since S_2 is an infinite surface, the vanishing of δI_1 requires that ϕ must be prescribed to an order of magnitude so that $\int_{S_2} \delta\phi\, \rho(\partial\phi/\partial n)\, ds = 0$. This however is not the case in fluid dynamics problems.

To clarify this point, let us consider the two-dimensional case. At infinity, the admitted velocity potential is required to behave as follows

$$\phi = Ur \cos\theta - \frac{K}{2\pi}\theta + (U + A_1)r^{-1} \cos\theta + A_2 r^{-1} \sin\theta + O(r^{-2}), \tag{6}$$

where r, θ are the polar coordinates, K is the strength of circulation A_1 and A_2 are to be determined and $O(r^{-2})$ represents terms of the order $1/r^2$ or higher. Thus

$$\delta\phi = \delta A_1 r^{-1} \cos\theta + \delta A_2 r^{-1} \sin\theta + O(r^{-2}),$$

where δA_1 and δA_2 are arbitrary. Writing ρ and $\partial\phi/\partial n$ in terms of ϕ as given in (6) and integrating, one obtains

$$\int_{s_2} \delta\phi\, \rho \frac{\partial\phi}{\partial n}\, ds = \int_{\theta=0}^{\theta=2\pi} \delta\phi\, \rho \frac{\partial \phi}{\partial n}\, r\, d\theta \Big]_{r=\infty} = -\rho_0 U \pi \delta A_1\,, \tag{7}$$

where ρ_0 is the density at infinity, s_1 is the boundary curve and ds is the elementary length. The vanishing of δI_1 then requires that $\rho_0 U$ must be zero because δA_1 is arbitrary. This however is not possible. It is therefore clear that Bateman's variational principle is not applicable to flows in which the domain extends to infinity. The appropriate principle in this case should be

$$\delta\left[\int_S p(\phi)\, dS + \rho_0 U \pi A_1\right] = 0, \tag{8}$$

where S is the surface of the domain and dS is the elementary surface.

3. A variational principle for steady, irrotational compressible flow with infinite domain. In applying the Rayleigh-Ritz method to the approximate solution of compressible

flow problems, it is found that for most problems the velocity potential ϕ may be written in the form

$$\phi = \phi_1 + \phi_2 , \tag{9}$$

where ϕ_1 denotes the velocity potential of the corresponding incompressible flow and ϕ_2 denotes the remaining part due to the compressibility effect. Substituting ϕ as given by (9) into Eq. (4) and carrying out the expansion, the expression for p may be written in the following form

$$p = A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k - \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i} + \cdots . \tag{10}$$

Since ϕ_1 is a definite function, $\delta\phi_1$ is zero, and hence

$$\delta \int_V \left[A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k\right] dV = 0, \tag{11}$$

$$\begin{aligned} \delta \int_V \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i}\, dV &= -\int_{S_1+S_2} \delta\phi_2\, \rho_0 \frac{\partial \phi_1}{\partial n}\, dS - \int_V \delta\phi_2\, \rho_0 \frac{\partial^2 \phi_1}{\partial x_i\, \partial x_i}\, dV \\ &= -\int_{S_2} \delta\phi_2\, \rho_0 \frac{\partial \phi_1}{\partial n}\, dS. \end{aligned} \tag{12}$$

The last step in (12) is obtained because ϕ_1 satisfied the Laplace equation $\partial^2\phi_1/\partial x_i \partial x_i = 0$ and on the stream surface S_1, $\partial\phi_1/\partial n = 0$ and thus $\int_{S_1} \delta\phi_2(\partial\phi_1/\partial n)\, dS = 0$.

As long as the Euler equation of a variational integral is not affected, it is permissible to change the original integral by adding or subtracting other integrals. Since (11) is zero and (12) gives only a boundary integral, the following variational integral will have the same Euler's equation as Bateman's integral (4).

$$I_2 = \int_V \left\{p(\phi) - \left[A + C\left(q_m^2 - \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_1}{\partial x_i}\right)^k\right]\right\} dV + \int_V \rho_0 \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_2}{\partial x_i}\, dV. \tag{13}$$

Noting that $[A + C(q_m^2 - (\partial\phi_1/\partial x_i)(\partial\phi_1/\partial x_i)^k] = p(\phi_1)$ and integrating the second integral in (13) by Green's formula, (13) becomes

$$I_2 = \int_V [p(\phi) - p(\phi_1)]\, dV - \int_{S_2} \phi_2\rho_0 \frac{\partial \phi_1}{\partial n}\, dS. \tag{14}$$

In the above expression, $p(\phi_1)$ subtracted thusly insures the boundedness of I_2. The last integral in (14) must be subtracted so that the boundary integral vanishes when the first variation of (14) is taken. In the two-dimensional case, the first variation of the last integral in (14) indeed reduces to (7).

In references [2]-[9], variational methods have been carried out to solve compressible flow problems following the Rayleigh-Ritz, the Galerkin, and Biezieno-Koch procedures. In all the problems solved, excellent results were obtained. In the case of the Galerkin method and the Biezieno-Koch method, the formulation of a variational principle is not necessary. However, in performing the numerical computation for potential flows past arbitrary bodies, it was found that the Rayleigh-Ritz method requires the least amount of labor.

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ON A GEOMETRICAL METHOD OF DERIVING THREE-DIMENSIONAL HARMONIC FLOWS FROM TWO-DIMENSIONAL ONES*

By AUREL WINTNER (*The Johns Hopkins University*)

The Flow Operators Ω. Let D be a domain in a (u,v)-plane, E a domain in an (x,y,z)-space, and let $\varphi = \varphi(u, v)$, $\psi = \psi(x, y, z)$ denote (real-valued and regular) solutions of $\Delta_2\varphi = 0$, $\Delta_3\psi = 0$ on D, E, respectively, where Δ_2 and Δ_3 denote the two- and three-dimensional euclidean Laplace operators,

$$\Delta_2 = \partial^2/\partial u^2 + \partial^2/\partial v^2 \qquad \text{and} \qquad \Delta_3 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \tag{1}$$

All harmonic functions φ are accessible in principle, since all of them are given by $\varphi(u, v) = Re\chi(w)$, where χ is any function which is regular-analytic in $w = u + iv$ on D. In contrast, there does not exist anything like this rule for the harmonic functions $\psi(x, y, z)$ on a three-dimensional E. Hence it is natural to ask for flow operators, say $\Omega = \Omega(D)$, which, from every regular solution $\varphi = \varphi(u, v)$ of $\Delta_2\varphi = 0$ on a two-dimensional (u,v)-domain D, will manufacture a regular solution,

$$\psi(x, y, z) = \Omega\varphi(u, v), \tag{2}$$

of $\Delta_3\psi = 0$ on a three-dimensional (x,y,z)-domain $E = E(D)$. The latter should not depend on the particular choice of the function $\varphi(u, v)$, but merely on the operator $\Omega = \Omega(D)$ and on the domain D on which $\varphi(u, v)$ is supposed to be harmonic.

A trivial instance of such "harmonic flow operators" Ω is supplied by the cylindrical flow which, from a given $\varphi(u, v)$, manufactures the corresponding $\psi(x, y, z)$ as follows:

$$\psi(x, y, z) = \varphi(x, y). \tag{3}$$

In fact, (3) is of the type (2), since $\Delta_3\psi(x, y, z) = \Delta_2\psi(x, y)$ if $\partial^2\psi/\partial z^2 = 0$; cf. (1). The

*Received June 30, 1950.

$E = E(D)$ belonging to the case (3) of (2) is the interior of the infinite cylinder having the (orthogonal) cross-section D.

The History of the Problem. It turns out that this trivial flow operator Ω, that defined by (3), is only the simplest instance of a large class of flow operators Ω which have the property specified in connection with (2). In fact, the general class of harmonic flow operators Ω was discovered by Weingarten sixty years ago.[1] His results are stated, however, in the language of the differential geometry of minimum surfaces, and seem to have remained unnoticed (except, possibly,[2] by F. Klein). In any case, the principal result of Weingarten was rediscovered by Levi-Civita a decade later,[3] by using, instead of the theory of minimum surfaces, extremely heavy explicit calculations, which he only was able to carry out due to Ricci's and his absolute differential calculus. In addition, these results of Levi-Civita are in a section which is not along the main lines of his long paper. Finally, the latter was published in a periodical not available in most libraries.

These circumstances notwithstanding, it is somewhat surprising that the Ω-flows in question seem to have been forgotten for half a century. On the other hand, the nature and the generality of the result appear to be elastic enough to admit its explicit use in solving certain three-dimensional boundary-value problems which could hardly be attacked in any other way. Under these circumstances, and because only the proof, but not the final result, is quite involved, the criterion in question seems to be worth recording here in a form which, it is hoped, can be followed by those who apply mathematics.

(i) *Line Congruences.* Let Γ be a 2-parameter family of straight lines which, with reference to a three-dimensional $(x,y,z,)$-domain E, have the following property: If P is any point of E, then Γ contains exactly one straight line, say $\gamma = \gamma_P$, through P. In the terminology of the differential geometry of "rays" γ, such a collection Γ is called a "line congruence" (on E). It will be assumed that that portion of each of the straight lines which is contained in E is a connected set (i.e., that it is a segment or a half-line, possibly the entire line γ).

Analytically, every sufficiently small portion of Γ can be described by first choosing inside E a surface, say S, and then introducing on S Gaussian parameters, say u and v, in an arbitrary way, subject only to the restriction that there be a one-to-one (and sufficiently differentiable) correspondence between the points of S and the points of a domain, say D, in a euclidean (u,v)-plane. In fact, if S and its (u,v)-parametrization are fixed, then a straight line γ belonging to Γ can be individualized by placing $\gamma = \gamma(u, v)$, where u, v are the parameter values of that point of S from which γ is issued.

(ii) *Rectilinear Flows.* In this manner, the line congruence Γ defines a rectilinear flow from the surface S into the three-dimensional (x,y,z)-domain E (which contains S).

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[1]J. Weingarten, Ueber particuläre Integrale der Differentialgleichung $\Delta V = 0$ und eine mit der Theorie der Minimalflächen zusammenhängende Gattung von Flüssigkeitsbewegungen, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1890, pp. 313-335.

[2]Cf. the footnote on p. 117 of Levi-Civita's paper, referred to in the next footnote. The common element in the treatments of Weingarten and Levi-Civita is Jacobi's theorem concerning the (complex-valued) solutions $f = f(x,y,z)$ of $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + (\partial f/\partial z)^2 = 0$, a theorem to which Levi-Civita's attention was called by Klein (cf. the footnote mentioned before).

[3]T. Levi-Civita, Tipi di potenziali che si possono far dipendere da due sole coordinate, Memorie della Reale Accademia delle Scienze de Torino, ser. 2, vol. 49 (1900), pp. 105-152; more particularly, pp. 138-139.

Correspondingly, if φ is any function of the position (u, v) on a two-dimensional domain, D, in the euclidean (u,v)-plane, with u, v as Gaussian parameters on S, then a function, say ψ, of the position (x, y, z) on E will be defined by placing

$$\psi(x, y, z) = \varphi(u, v) \text{ if } (x, y, z) \text{ is on } \gamma(u, v). \tag{4}$$

It is understood that $\gamma(u, v)$ in (4) denotes that straight line, γ, of Γ which reaches the surface S at the point having the Gaussian coordinates (u, v).

(iii) *Laplacean Rectilinear Flows.* Clearly, the operation (4) is of the type (2). But it was not assumed in (4) that the given function, $\varphi(u, v)$, satisfies $\Delta_2\varphi = 0$. Moreover, if $\Delta_2\varphi = 0$ is assumed, it is not in general true that the function $\psi(x, y, z)$ defined by (4) will satisfy $\Delta_3\psi = 0$. The operators Δ_2, Δ_3 are those defined by (1).

This problem is solved by the result of Weingarten and Levi-Civita, referred to above. It states that, if $\varphi(u, v)$ is *any* (continuous) solution of $\Delta_2\varphi = 0$ on a (u,v)-domain D, then the function $\psi(x, y, z)$ defined by (4) will be a solution of $\Delta_3\psi = 0$ on the corresponding (x,y,z)-domain $E = E_D$, provided that the rectilinear flow, on which the assignment (4) is based, is derived from a line congruence Γ which is an isotropic line congruence.

The latter notion, to be defined under (iv) below, depends only on the collection of lines which constitute Γ, rather than on the particular choice of the surface S and of the Gaussian parametrization (u, v) of S. Consequently, neither of these choices matters, even though both of them occur in (4).

(iv) *Isotropy.* There remains to be defined the notion of an isotropic line congruence, mentioned above. The customary definition of this notion, introduced by Ribacour,[4] is quite involved. In what follows, it will be replaced by another definition, one which can be stated more easily and which, as a matter of proof,[5] turns out to be equivalent to the usual definition.

Consider in the (x,y,z)-space two (sufficiently small and sufficiently differentiable) pieces of surfaces, say S and T, and let $P \to Q = Q_P$ be a one-to-one (and sufficiently differentiable) mapping of the points, P, of S on the points, Q, of T. Suppose that every S-point, P, is distinct from its T-image, $Q = Q_P$, i.e., that every pair P, Q determines a straight line $\gamma = \gamma(P; Q)$ and therefore the entire mapping, $S \to T$, defines a certain line congruence, say Γ. Suppose further that the mapping $S \to T$ defining this Γ has the following two properties:

(α) $P \to Q = Q_P$ is an isometric mapping of S on T (as to this notion, cf. (v) below) and

(β) $|PQ| =$ const., i.e., the euclidean length of the segment joining P with $Q = Q_P$ (in the (x,y,z)-space) is independent of the choice of P on S.

If these conditions are satisfied by the mapping $S \to T$, then the line congruence Γ, mentioned before (α), is an isotropic line congruence. Conversely, every isotropic line congruence can be obtained in this manner, that is, by suitably choosing two surfaces, S and T, and a mapping, $S \to T$, which satisfies both of the above conditions, (α) and (β).

(v) *Isometry.* For the sake of completeness, the notion of isometry, occurring above in (α), remains to be explained. This notion is familiar from the elements of the theory of surfaces. It is defined as follows:

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[4]L. Bianchi, Lezioni di geometria differenziale, 2nd edition, vol. 1 (1902), p. 302.

[5]Loc. cit.,[4] p. 304.

A one-to-one mapping of the points, P, of the surface, S, on the points, $Q = Q_P$, of another surface, T, is called isometric if, by virtue of the mapping, the squared line-element, ds^2, on S (i.e., the so-called "metric" or "first fundamental form", $g_{ik}\, du^i\, du^k$, on S) coincides with the squared line-element on T.

In order that this be the case, it is necessary that the Gaussian curvature of S at any point P be identical with the Gaussian curvature of T at the corresponding point $Q = Q_P$. This necessary condition is sufficient as well in the particular case of surfaces of constant Gaussian curvature. Since cylinders are of constant Gaussian curvature, the constant being 0, this embeds the trivial example (3) into the general theory.

FREE LONGITUDINAL VIBRATION OF A PROLATE ELLIPSOID, CLAMPED CENTRALLY*

BY JAMES S. KOUVELITES (*Sloane Physics Laboratory, Yale University***)

Introduction. It was originally shown by Poisson[1] that the magnetization is uniform and parallel throughout the interior of an ellipsoid of magnetically isotropic material placed in a previously uniform and parallel static field. Since this unique property of the ellipsoid allows an exact calculation of its demagnetizing factor as well as a simple field analysis, the prolate spheroid was chosen as the appropriate shape for the specimens whose magnetostrictive vibrational properties are being studied in this laboratory.[2] Although the magnetostrictive vibration is forced rather than free, the damping factor has been found to be so small for the materials tested, that through an investigation of the free vibration, a reasonably good approximation to some of the resonance phenomena may be obtained.

Analysis and discussion. For later comparison with the corresponding expressions for the ellipsoid, the differential equation of motion, its integrated solution, and a subsidiary expression for the frequencies of resonance for free longitudinal vibration of a bar of constant cross-section, clamped centrally,[3] are

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{k} \frac{\partial^2 \xi}{\partial t^2} \tag{1}$$

$$\xi = A \sin\left(\omega\left(\frac{\rho}{k}\right)^{1/2} x\right) \sin(\omega t + \phi) \tag{2}$$

$$f_{R_n} = \frac{n}{4a}\left(\frac{k}{\rho}\right)^{1/2}, \qquad n = 1, 3, 5, \cdots \tag{3}$$

*Received Sept. 26, 1950.

**Assisted by the Office of Naval Research.

[1]As described in J. C. Maxwell, *A treatise on electricity and magnetism*, vol. II, 3rd ed., Clarendon Press, Oxford, 1892, pp. 66-69.

[2]J. S. Kouvelites and L. W. McKeehan, *Magnetostrictive vibration of prolate spheroids. Preliminary measurements*, Rev. Sci. Instr., in press.

[3]Rayleigh, *Theory of sound*, vol. I, §150.

where x is the coordinate along the longitudinal axis (the origin being at the center of the bar), t the time, ξ the longitudinal displacement, ρ the volume density, k the coefficient of elasticity, a the half-length of the bar, ω the angular frequency, A and ϕ arbitrary amplitude and phase constants.

For an elastically-isotropic prolate ellipsoid with two eccentricities nearly equal to unity as in the physical problem of magnetostrictive vibration, it appears reasonable to assume that the equiphase surfaces of ξ are planes perpendicular to the major axis.[4] Applying the second law of motion to a transverse plane lamina of thickness dx of an ellipsoid bounded by the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad \text{where } 2a \text{ is the major axis,}$$

$$\frac{\partial F}{\partial x}\, dx = \frac{\partial^2 \xi}{\partial t^2}\, dm, \qquad \text{where } dm = \pi y z \rho\, dx = \pi bc\left(1 - \frac{x^2}{a^2}\right)\rho\, dx.$$

Since Hooke's law may be stated as

$$F = \pi bc\left(1 - \frac{x^2}{a^2}\right)k\,\frac{\partial \xi}{\partial x},$$

the equation of motion becomes

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{2x}{a^2 - x^2}\frac{\partial \xi}{\partial x} = \frac{\rho}{k}\frac{\partial^2 \xi}{\partial t^2}, \tag{4}$$

which is a definite departure from the familiar wave equation (1) appropriate to a channel of uniform cross section.

The separation of the variables, x and t, is accomplished as usual by letting $\xi = X(x)T(t)$, which is substituted into equation (4). Both sides of the resultant expression may then be set equal to the constant, $-\omega^2\rho/k$, yielding:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{5}$$

$$\frac{d^2X}{dx^2} - \frac{2x}{a^2 - x^2}\frac{dX}{dx} + \omega^2\frac{\rho}{k}X = 0 \tag{6}$$

Thus, $\xi = X \sin(\omega t + \phi)$ where X may be obtained as an explicit function of x by inserting it into Eq. (6) in the form of the power series,

$$X = A_0\left(\frac{x}{a}\right)^p + A_1\left(\frac{x}{a}\right)^{p+1} + \cdots + A_s\left(\frac{x}{a}\right)^{p+s} + \cdots \tag{7}$$

As a result, it is found that, if $p = 0$, both A_0 and A_1 have arbitrary, finite values and the coefficients of Eq. (7) satisfy the condition

$$A_{s+2} = \frac{s}{s+2}A_s + \frac{N}{(s+1)(s+2)}(A_{s-2} - A_s), \tag{8}$$

$$\text{where } N = \omega^2 a^2 \rho/k$$

[4]See Rayleigh[3], Vol. II, §265 and P. M. Morse, *Vibration and sound*, McGraw-Hill Book Co., New York, 1936, pp. 216-217 for analogous treatment of sound transmission in tubes of varying cross-section.

Moreover, since the physical situation requires that the ellipsoid be clamped centrally (i.e. $\xi = 0$ at $x = 0$ at any time), A_0, A_2, A_4, etc., must be zero. The series solution for X, subject to Eq. (8), then becomes

$$X = A_1\left(\frac{x}{a}\right) + A_3\left(\frac{x}{a}\right)^3 + A_5\left(\frac{x}{a}\right)^5 + \cdots . \tag{9}$$

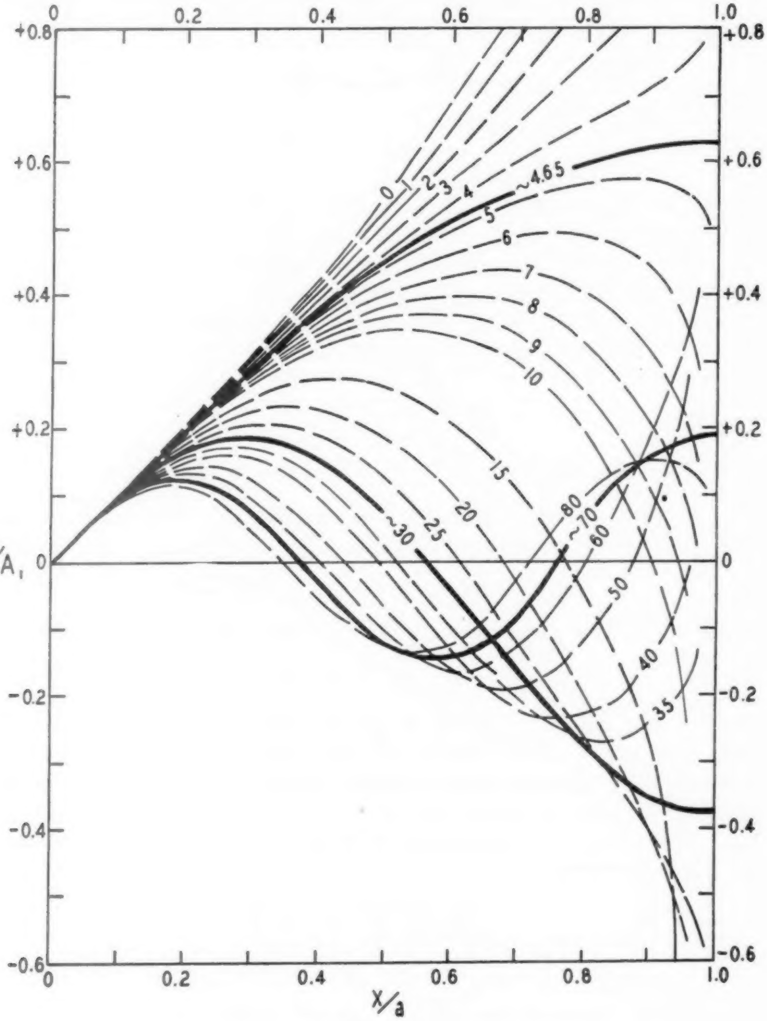

Fig. 1.

There is no force exerted on the ends of the ellipsoid. Consequently, the second boundary condition of the problem is that $dX/dx = 0$ at $x = \pm a$. The particular values

of N which satisfy this condition, are found by a trial and error process from Eqs. (8) and (9). It is interesting to note that Eq. (9), for $x = \pm a$, is divergent for all values of N except those corresponding to the modes of vibration satisfying the second boundary condition.

In Fig. 1, X/A_1, has been plotted as a function of x/a for various values of N, the thickly-drawn curves representing the "standing-wave" patterns for the fundamental mode and its first two overtones. It was sufficient to show the deflection variations over only one half the length of the ellipsoid because there is symmetry about the plane $x = 0$.

The frequencies of resonance for the ellipsoid may thus be expressed as

$$f_{R_n} = \frac{n}{4a}\left(\frac{k}{\rho}\right)^{1/2} \qquad \text{where } n \simeq 1.38,\ 3.49,\ 5.33,\ \cdots, \tag{10}$$

indicating the inharmonic nature of the overtones. However, a comparison between Eqs. (3) and (10) reveals that the consecutive overtones of the vibrating ellipsoid have ratios closer and closer to the ratios of adjacent odd integers as the value of n increases.

Acknowledgements. I wish to thank Professor L. W. McKeehan of this laboratory, whom I am assisting in the study of magnetostrictive vibration, and Professor F. J. Beck, Jr., of the Dunham Laboratory of Electrical Engineering, for encouraging discussions of this problem.

THE LEAST SQUARES SOLUTION FOR A SET OF COMPLEX LINEAR EQUATIONS*

By R. TURETSKY (*Aberdeen Proving Ground*)

Consider the set of m observational equations whose matrix representation is

$$Az \sim w, \tag{1}$$

where A is an $m \times n$ $(m \geq n)$ matrix of rank n whose elements are prescribed complex quantities, while w is an $m \times 1$ matrix. We seek that z (an $n \times 1$ matrix) which minimizes the sum of the squares of the absolute values of the components of the vector $Az - w$.

Set $A = B + iC$, $z = x + iy$, $w = u + iv$, where B, C, x, y, u, and v are real matrices. Then Eq. (1) is equivalent to

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

This is the matrix representation for a set of real observational equations. To obtain the normal equations,[1] we multiply on the left by the transpose of the coefficient matrix

*Received July 28, 1950.

[1]See Whittaker and Robinson, *Calculus of observations*, Chapter IX.

of the unknowns. Letting primes denote the transpose, there results

$$\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B' & C' \\ -C' & B' \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3}$$

where

$$P = B'B + C'C$$

and

$$Q = C'B - B'C \tag{4}$$

It will be noted that P is symmetric while Q is skew-symmetric, which greatly facilitates the computation.

The solution of Eq. (3) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}^{-1}\begin{pmatrix} B' & C' \\ C' & B' \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{5}$$

Ordinarily the inverse of the $2n \times 2n$ matrix

$$\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$$

is obtained directly. However, recourse may be had to its special structure to note that the inverse is of the form

$$\begin{pmatrix} R & S \\ -S & R \end{pmatrix}$$

where

$$R = (P + QP^{-1}Q)^{-1}$$

$$S = P^{-1}QR \tag{6}$$

The computation is thus simplified, since we have to compute the inverses of two $n \times n$ matrices rather than that of one $2n \times 2n$ matrix. Moreover, advantage can be taken of the fact that R is symmetric while S is skew-symmetric.

Finally, the multiplication of

$$\begin{pmatrix} R & S \\ -S & R \end{pmatrix}$$

by

$$\begin{pmatrix} B' & C' \\ -C' & B' \end{pmatrix}$$

required on the right hand side of Eq. (5) leads to the matrix

$$\begin{pmatrix} T & -U \\ U & T \end{pmatrix}$$

where

$$\begin{aligned} T &= RB' - sC' \\ U &= RC' + sB' \end{aligned} \tag{7}$$

Multiplication by a $2m \times 2n$ matrix is thus required only in the final step.

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